Linear Algebra Primer

Note: the slides are based on CS131 (Juan Carlos et al) and EE263 (by Stephen Boyd et al) at Stanford. Reorganized, revised, and typed by Hao Su

Outline

- Vectors and Matrices
  - Basic matrix operations
  - Determinants, norms, trace
  - Special matrices
- Transformation Matrices
  - Homogeneous matrices
  - Translation
- Matrix inverse
- Matrix rank

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Vector

- A column vector $v \in \mathbb{R}^{n \times 1}$ where

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $v^T \in \mathbb{R}^{1 \times n}$ where

$$v^T = [v_1 v_2 \ldots v_n]$$

T denotes the **transpose** operation

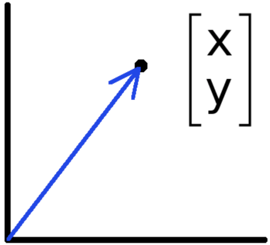
Vector

- We'll default to column vectors in this class

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- You'll want to keep track of the orientation of your vectors when programming in Python

Vectors have two main uses



- Vectors can represent an offset in 2D or 3D space
- Points are just vectors from the origin

- Data (pixels, gradients at an image keypoint, etc) can also be treated as a vector
- Such vectors do not have a geometric interpretation, but calculations like "distance" can still have value

Matrix

- A matrix $A \in \mathbb{R}^{m \times n}$ is an array of numbers with size m by n, i.e., m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

- if $m = n$, we say that A is square.

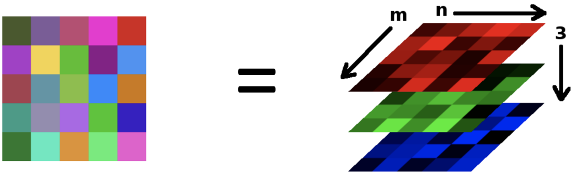
Images

$$= \begin{bmatrix} 193 & 180 & 210 & 112 & 125 \\ 189 & 8 & 177 & 97 & 114 \\ 100 & 71 & 81 & 195 & 165 \\ 167 & 12 & 242 & 203 & 181 \\ 44 & 25 & 9 & 48 & 192 \end{bmatrix}$$

- Python represents an image as a matrix of pixel brightness
- Note that the upper left corner is $(y, x) = [0, 0]$

Color Images

- Grayscale images have one number per pixel, and are stored as an $m \times n$ matrix
- Color images have 3 numbers per pixel – red, green, and blue brightness (RGB)
- stored as an $m \times n \times 3$ matrix



Basic Matrix Operations

We will discuss:

- Addition
- Scaling
- Dot product
- Multiplication
- Transpose
- Inverse/pseudo-inverse
- Determinant/trace

Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

  - Can only add a matrix with matching dimensions or a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

Vectors

- Norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- More formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 proerties:
  - Non-Negativity: For all $x \in \mathbb{R}^n$, $f(x) \geq 0$
  - Definiteness: $f(x) = 0$ if and only if $x = 0$
  - Homogeneity: For all $x\mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t|f(x)$
  - Triangle inequality: For all $x, y \in \mathbb{R}^n$, $f(x + y) \leq f(x) + f(y)$

Vector Operations

- Example norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|_\infty \qquad \|x\|_\infty = \max_i |x_i|$$

- General ℓ_p norms:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

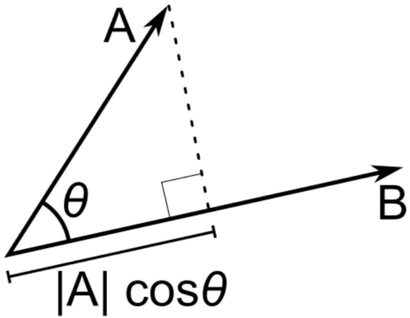
Vector Operations

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - $x \cdot y$ is also $|x||y|\cos(\text{the angel between } x \text{ and } y)$

$$x^T y = [x_1 \dots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

Vector Operations

- Inner product (dot product) of vectors
  - If B is a unit vector, then $A \cdot B$ gives the length of A, which lies in the direction of B



Matrix Operations

- The product of two matrices

$$A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

$$C_{ij} = \sum_{i=1}^{n} A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_m^T- \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

Matrix Operations

Multiplication example:

A x B

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 10 & 14 \\ 34 & 54 \end{bmatrix}$$

$$0 \cdot 3 + 2 \cdot 7 = 14$$

Each entry of the matrix product is made by taking the dot product of the corresponding row in the left matrix, with the corresponding column in the right one.

Matrix Operations

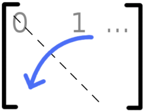
- The product of two matrices
  Matrix multiplication is associative: (AB)C=A(BC)
  Matrix multiplication is distributive: A(B+C)=AB+AC
  Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$ (For example, if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times q}$, the matrix product BA does not even exist if m and q are not equal!)

Matrix Operations

- Powers
  - By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
  - Obviously only square matrices can be multiplied that way

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

Matrix Operations

- Determinant
  - $\det(A)$ returns a scalar
  - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
  - For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$
  - Properties:

$$
\begin{aligned}
\det(AB) &= \det(A)\det(B) \\
\det(AB) &= \det(BA) \\
\det(A^{-1}) &= \frac{1}{\det(A)} \\
\det(A^T) &= \det(A) \\
\det(A) = 0 &\iff A \text{ is singular}
\end{aligned}
$$

Matrix Operations

- Trace
  - $\text{trace}(A) =$ sum of diagonal elements

$$\text{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  - Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely used in this class, though)
  - Properties:

$$\text{tr}(AB) = \text{tr}(BA)$$
$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$
$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$$

Matrix Operations

- Vector norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- Matrix norms: Norms can also be defined for matrices, such as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)}$$

Special Matrices

- Identity matrix I

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Special Matrices

- Symmetric matrix: $A^T = A$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix: $A^T = -A$

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$

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Transformation

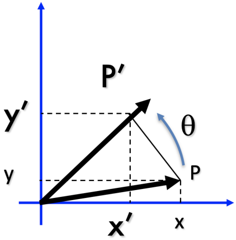
- Matrices can be used to transform vectors in useful ways, through multiplication: $x' = Ax$
- Simplest is scaling:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

(Verify by yourself that the matrix multiplication works out this way)

Rotation (2D case)

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta x - \sin\theta y$$
$$y' = \cos\theta y + \sin\theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = RP$$

Transformation Matrices

- Multiple transformation matrices can be used to transform a point:

$$p' = R_2 R_1 S p$$

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$$(R_2(R_1(Sp)))$$

Transformation Matrices

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- The effect of this is to apply their transformations one after the other, from **right to left**
- In the example above, the result is

$$(R_2(R_1(Sp)))$$

- The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:

$$p' = (R_2 R_1 S)p$$

Homogeneous System

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

  - This is sufficient for scale, rotate, skew transformations
  - But notice, we cannot add a constant! :(

Homogeneous System

- The (somewhat hacky) solution? Stick a "1" at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale, and skew like before, AND translate (note how the multiplication works out, above)
- This is called "homogeneous coordinates"

Homogeneous System

- In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

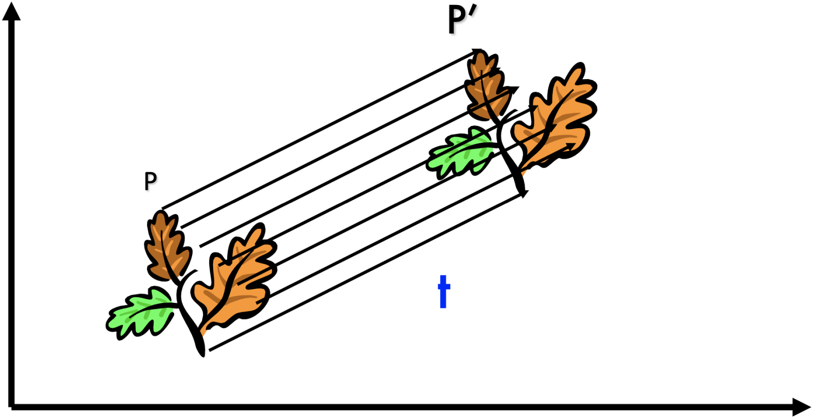
- Generally, a homogeneous transformation matrix will have a bottom row of [0 0 1], so that the result has a "1" at the bottom, too.

Homogeneous System

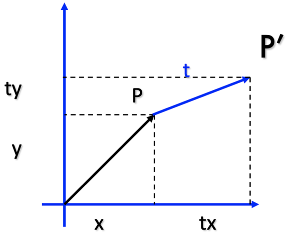
- One more thing we might want: to divide the result by something:
  - Matrix multiplication cannot actually divide
  - So, **by convention**, in homogeneous coordinates, we'll divide the result by its last coordinate after doing a matrix multiplication

$$\begin{bmatrix} x \\ y \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x/7 \\ y/7 \\ 1 \end{bmatrix}$$

2D Transformation using Homogeneous Coordinates



2D Transformation using Homogeneous Coordinates



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

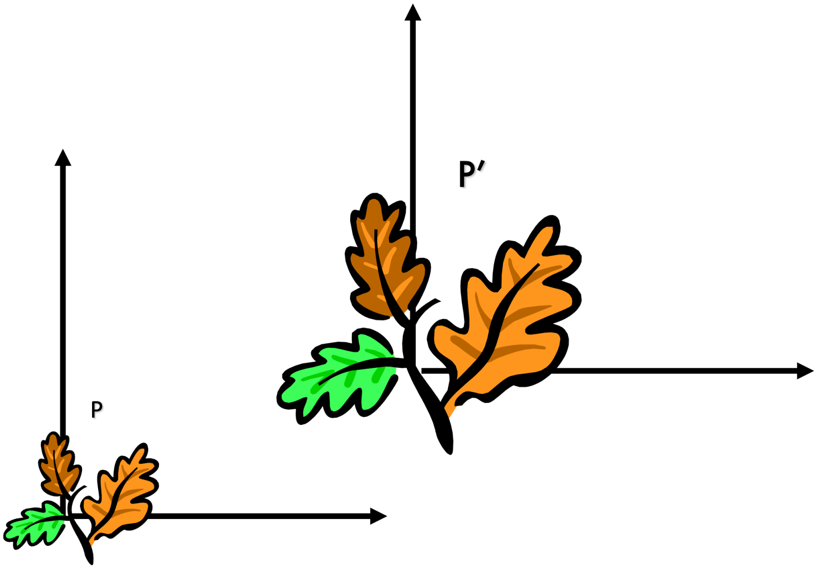
t

P

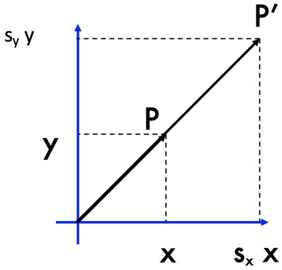
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



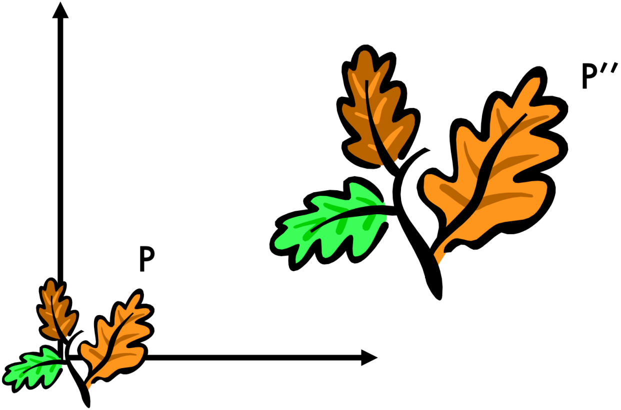
$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P'} = (\mathrm{s_x x}, \mathrm{s_y y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P'} = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



$$P'' = T \cdot P' = T \cdot (S \cdot P) = T \cdot S \cdot P$$

Scaling & Translating

$$P'' = T \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation & Scaling versus Scaling & Translating

$$
\begin{aligned}
P''' = T \cdot S \cdot P &= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}
\end{aligned}
$$

Translation & Scaling $\neq$ Scaling & Translating

$$
\begin{aligned}
P''' = T \cdot S \cdot P &= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
P''' = S \cdot T \cdot P = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =
$$

Translation & Scaling $\neq$ Scaling & Translating

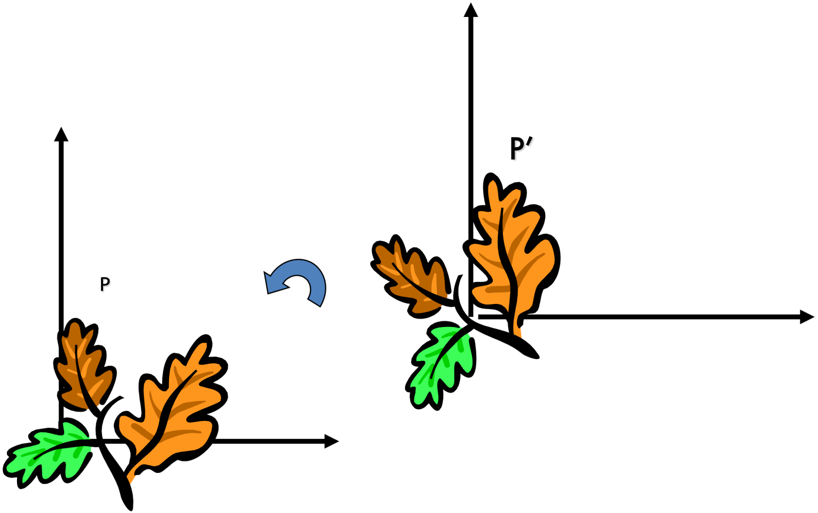
$$P''' = T \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$P''' = S \cdot T \cdot P = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

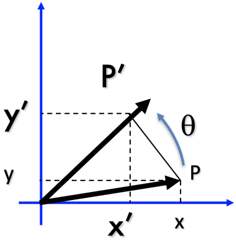
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta x - \sin\theta y$$
$$y' = \cos\theta y + \sin\theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = RP$$

Rotation Matrix Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix 2×2

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$R \cdot R^T = R^T \cdot R = I$$

$$\det(R) = 1$$

Rotation Matrix Properties

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$R \cdot R^T = R^T \cdot R = I$$
$$\det(R) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
  - (and so are its columns)

Scaling+Rotation+Translation

$$P' = (T\ R\ S)\ P$$

$$P' = T \cdot R \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} RS & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Outline

Inverse

- Given a matrix A, its inverse A^{-1} is a matrix such that $AA^{-1} = A^{-1}A = I$
- e.g.,

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

- Inverse does not always exist. If A^{-1} exists, A is *invertible* or *non-singular*. Otherwise, it is *singular*.
- Useful identities, for matrices that are invertible:

$$\begin{aligned} (A^{-1})^{-1} &= A \\ (AB)^{-1} &= B^{-1}A^{-1} \\ A^{-T} &\triangleq (A^T)^{-1} = (A^{-1})^T \end{aligned}$$

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Linear Independence

- Suppose we have a set of vectors $v_1, \ldots, v_n$
- If we can express v_1 as a linear combination of the other vectors $v_2, \ldots, v_n$, then v_1 is linearly *dependent* on the other vectors
  - The direction v_1 can be expressed as a combination of the directions $v_2, \ldots, v_n$ (e.g., $v_1 = 0.7v_2 - 0.7v_4$)

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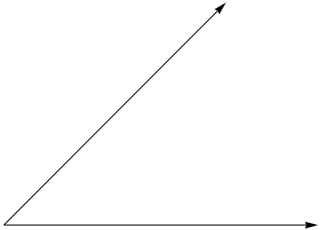
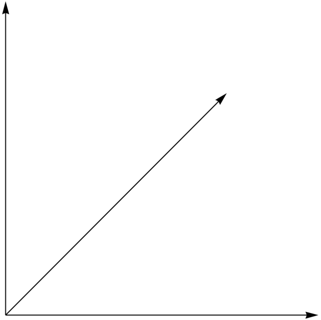
  - The direction v_1 can be expressed as a combination of the directions $v_2, \ldots, v_n$ (e.g., $v_1 = 0.7v_2 - 0.7v_4$)
- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
  - Common case: a set of vectors $v_1, \ldots, v_n$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero).

Linear Independence

Linearly independent set



Not linearly independent



Matrix Rank

- Column/row rank

$$\text{col-rank}(A) = \text{the maximum number of linearly independent column vectors of } A$$
$$\text{row-rank}(A) = \text{the maximum number of linearly independent row vectors of } A$$

  - Column rank always equals row rank
- Matrix rank

$$\text{rank}(A) \triangleq \text{col-rank}(A) = \text{row-rank}(A)$$

Matrix Rank

- For transformation matrices, the rank tells you the dimensions of the output
- e.g. if rank of A is 1, then the transformation

$$p' = Ap$$

  maps points onto a line.
- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 2y \end{bmatrix}$$

Matrix Rank

- If an $m \times m$ matrix is rank m, we say it is "full rank"
  - Maps an $m \times 1$ vector uniquely to another $m \times 1$ vector
  - An inverse matrix can be found
- If rank $<$ m, we say it is "singular"
  - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
  - Inverse does not exist
- Inverse also does not exist for non-square matrices