CSE 152: Computer Vision

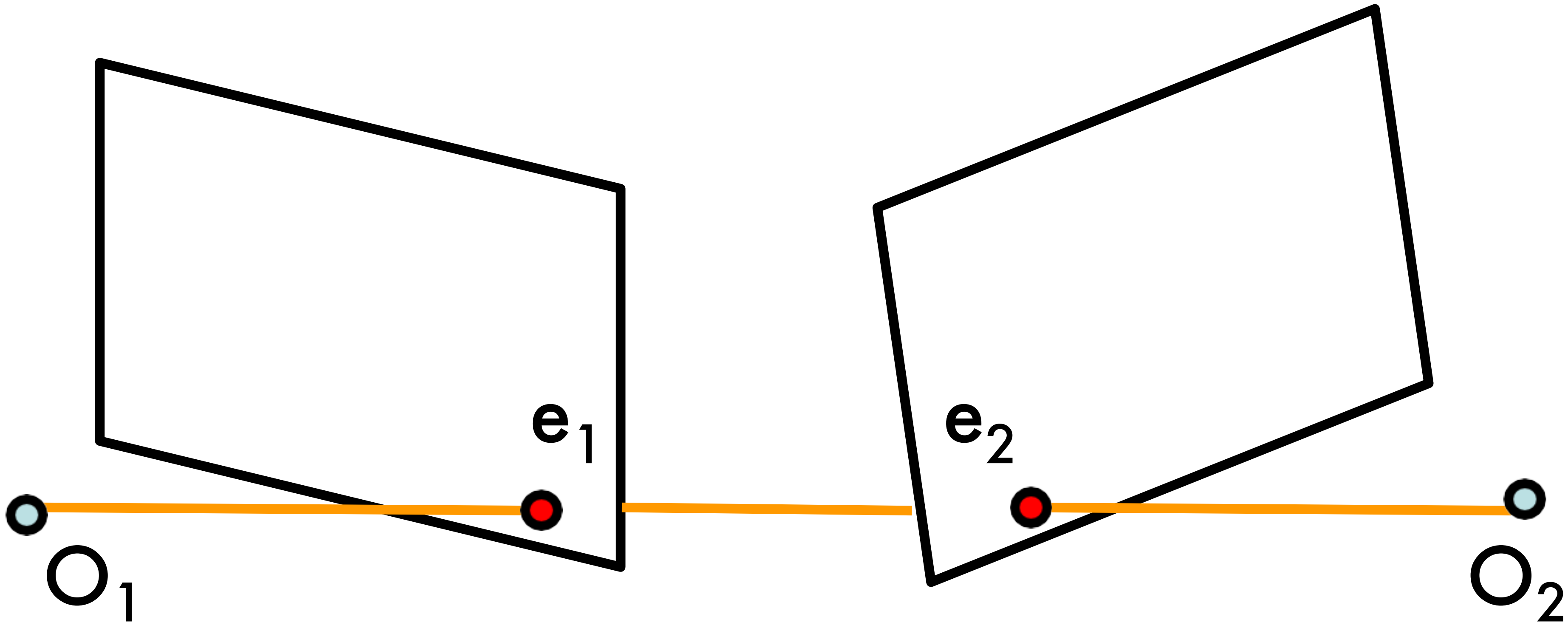
Hao Su

Lecture 15: Fundamental Matrix

Agenda

- **Review: Epipolar Geometry**
- Fundamental matrix
- Estimating F

Epipolar Geometry

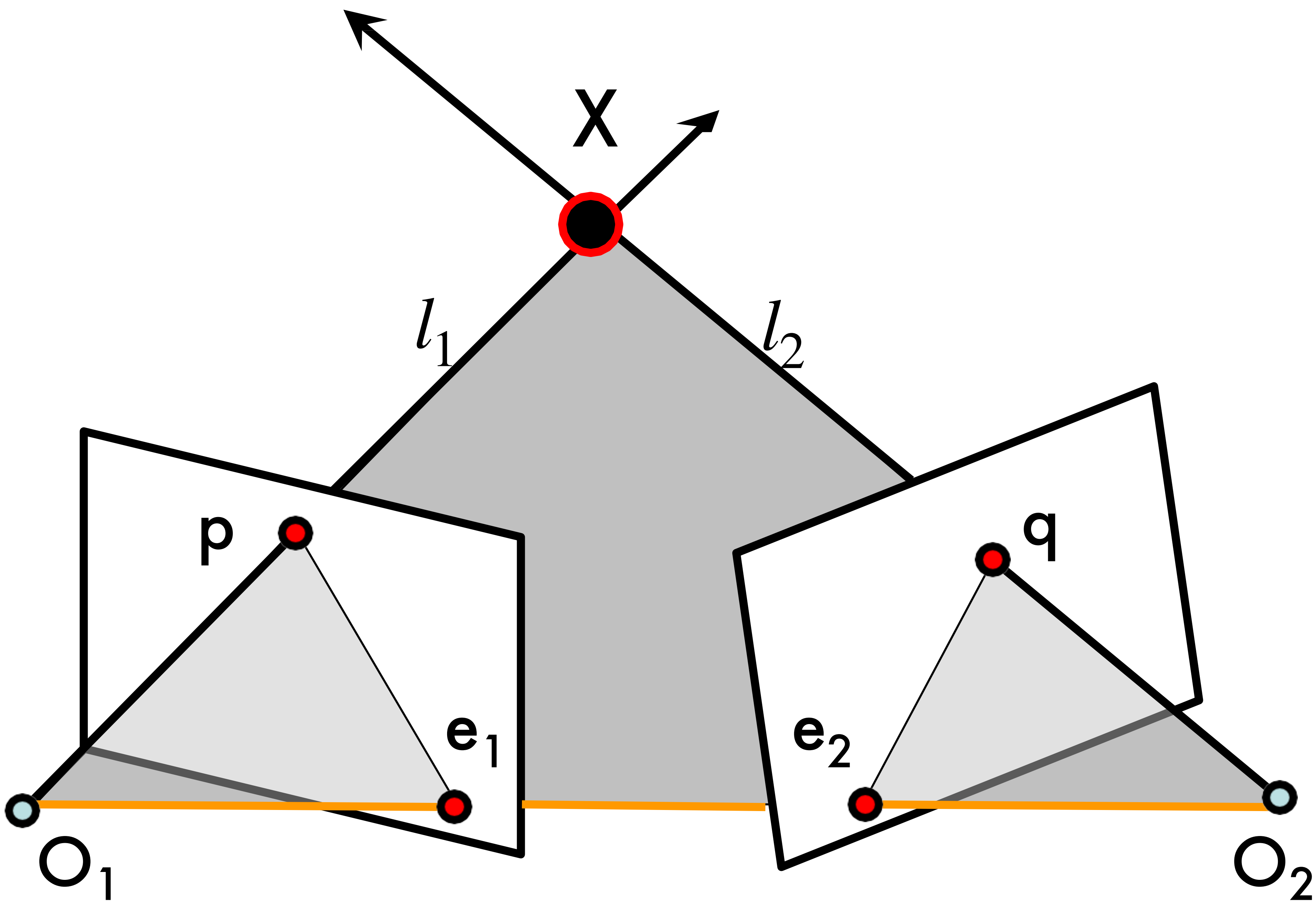


- Baselines
- Epipoles: e_1 , e_2

  = intersections of baseline with image planes

  = projections of the other camera center

Epipolar Geometry



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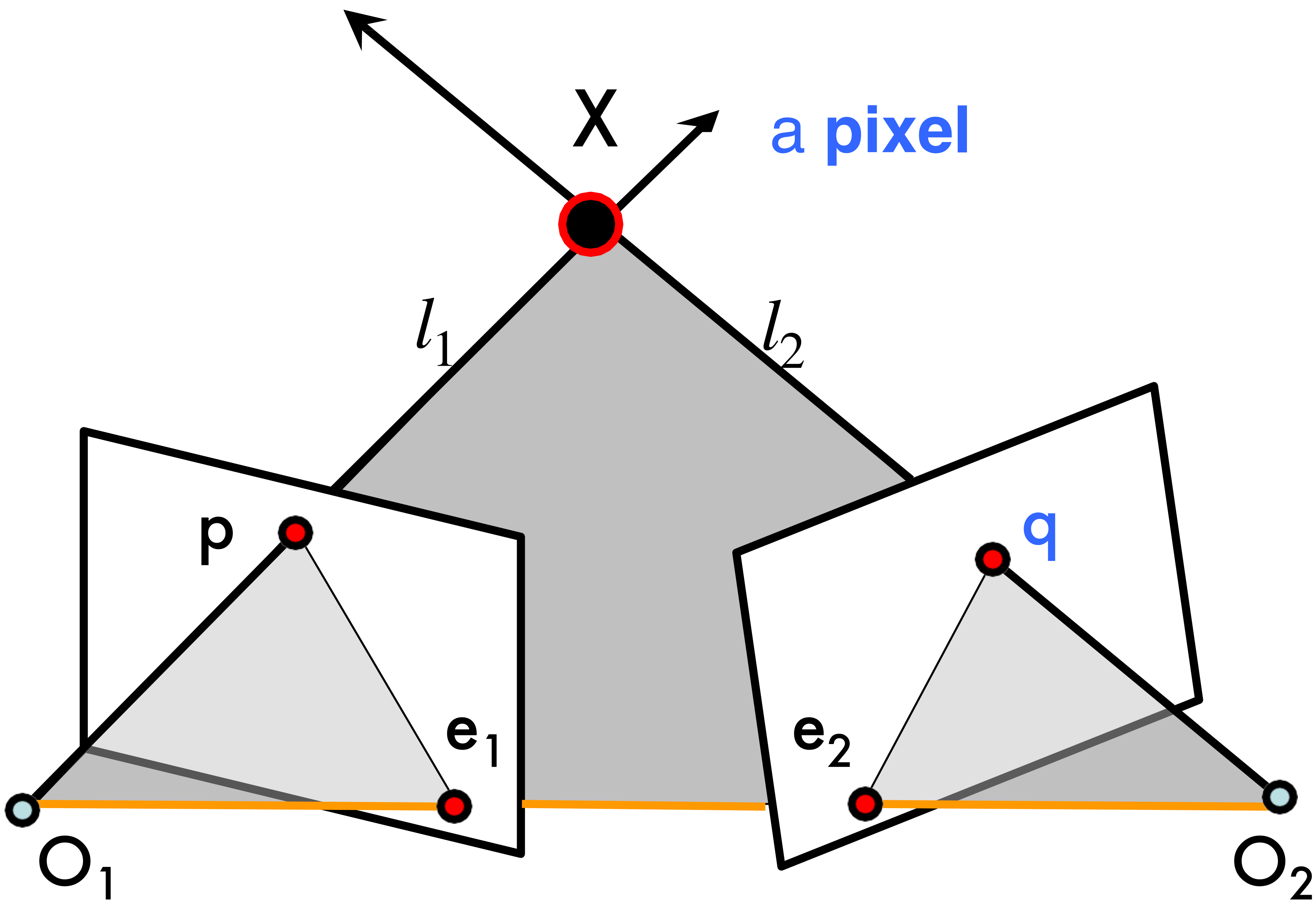

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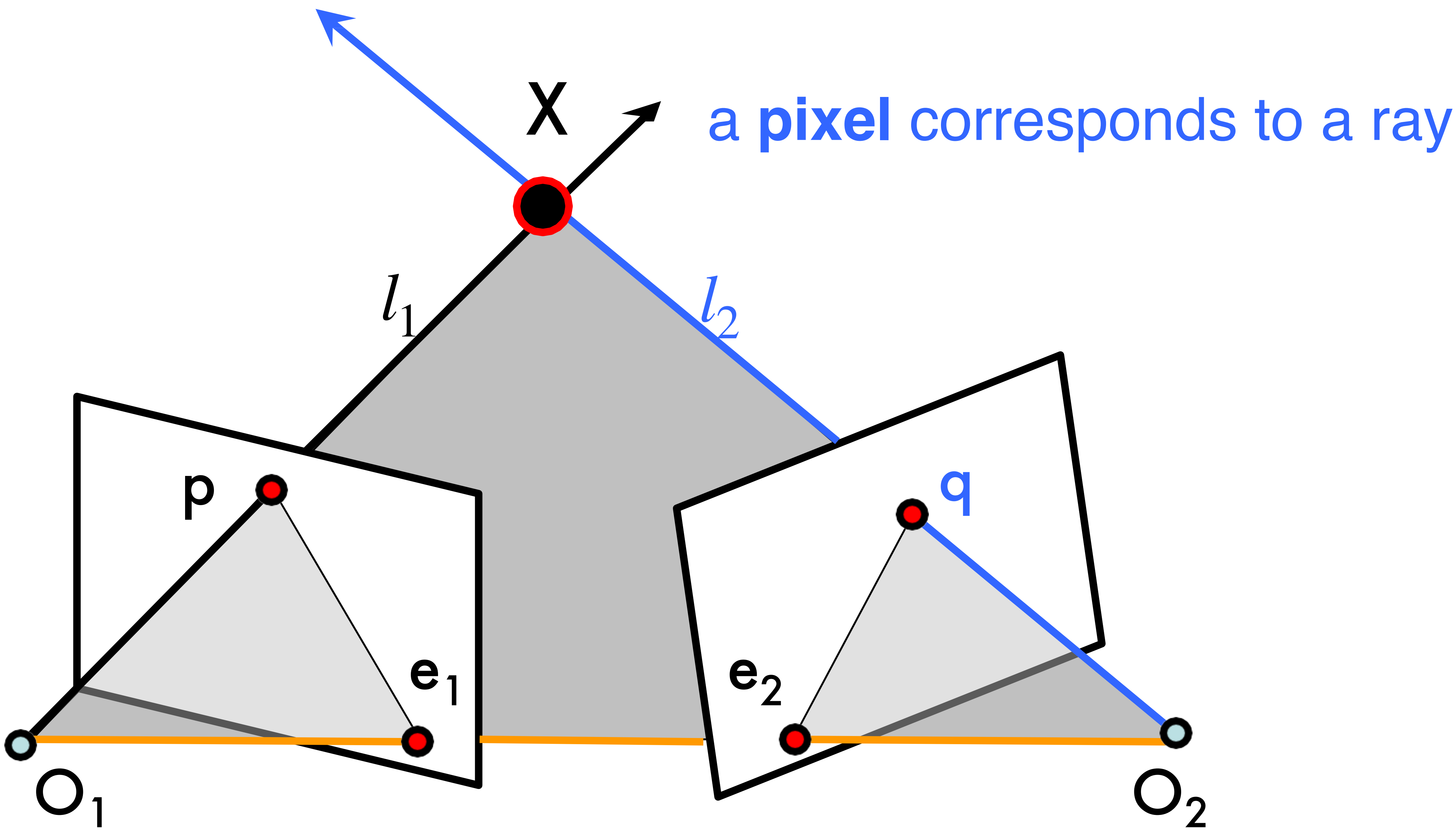

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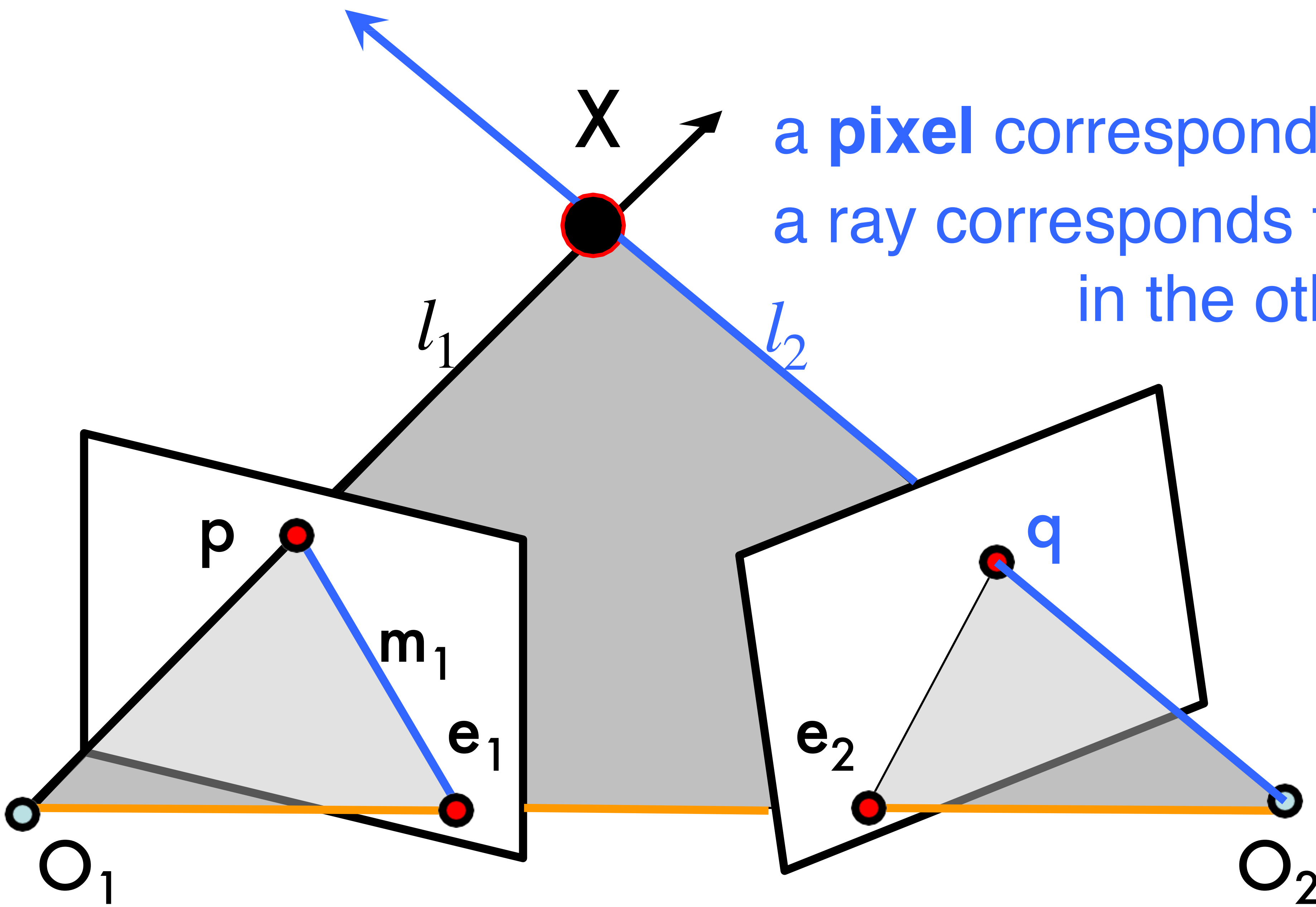

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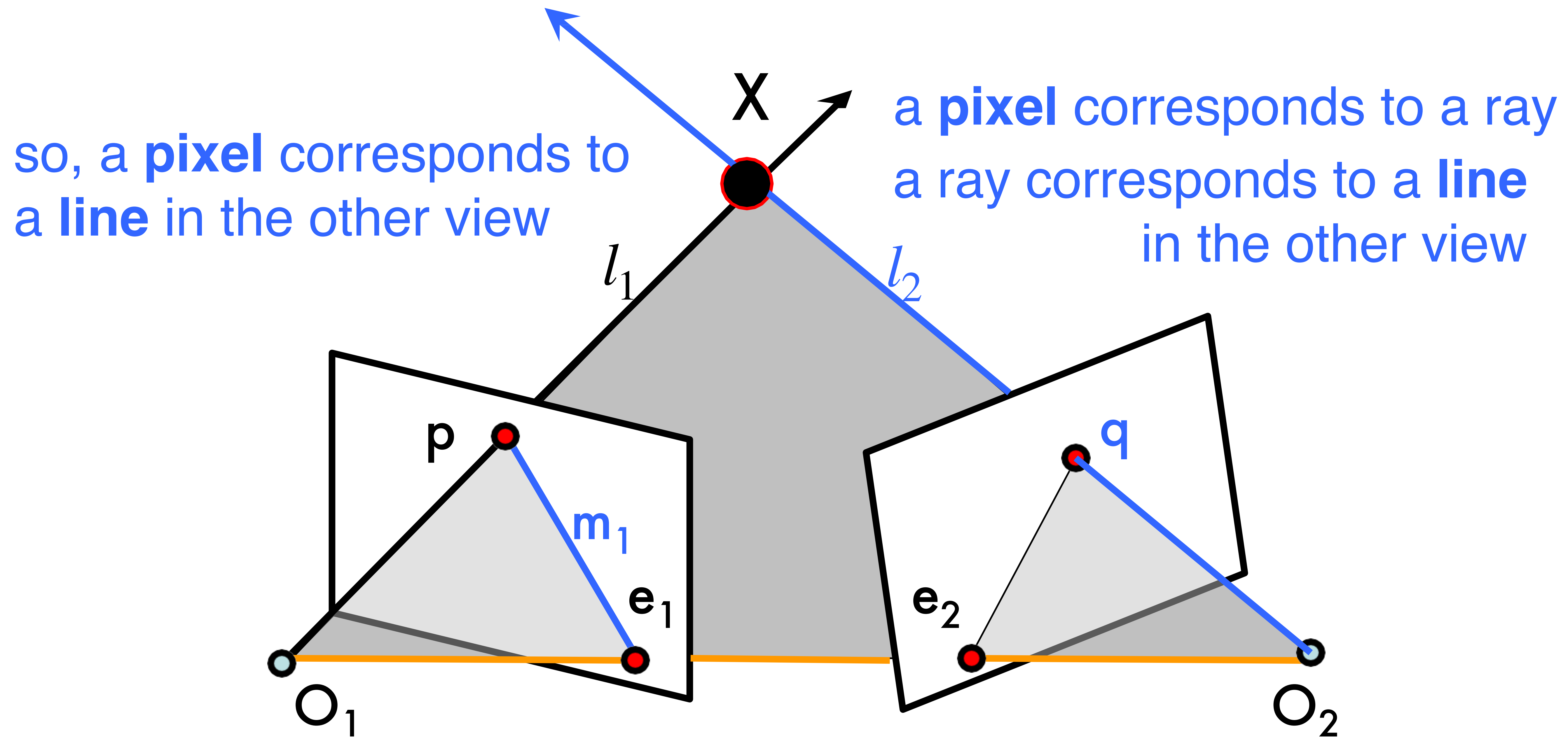

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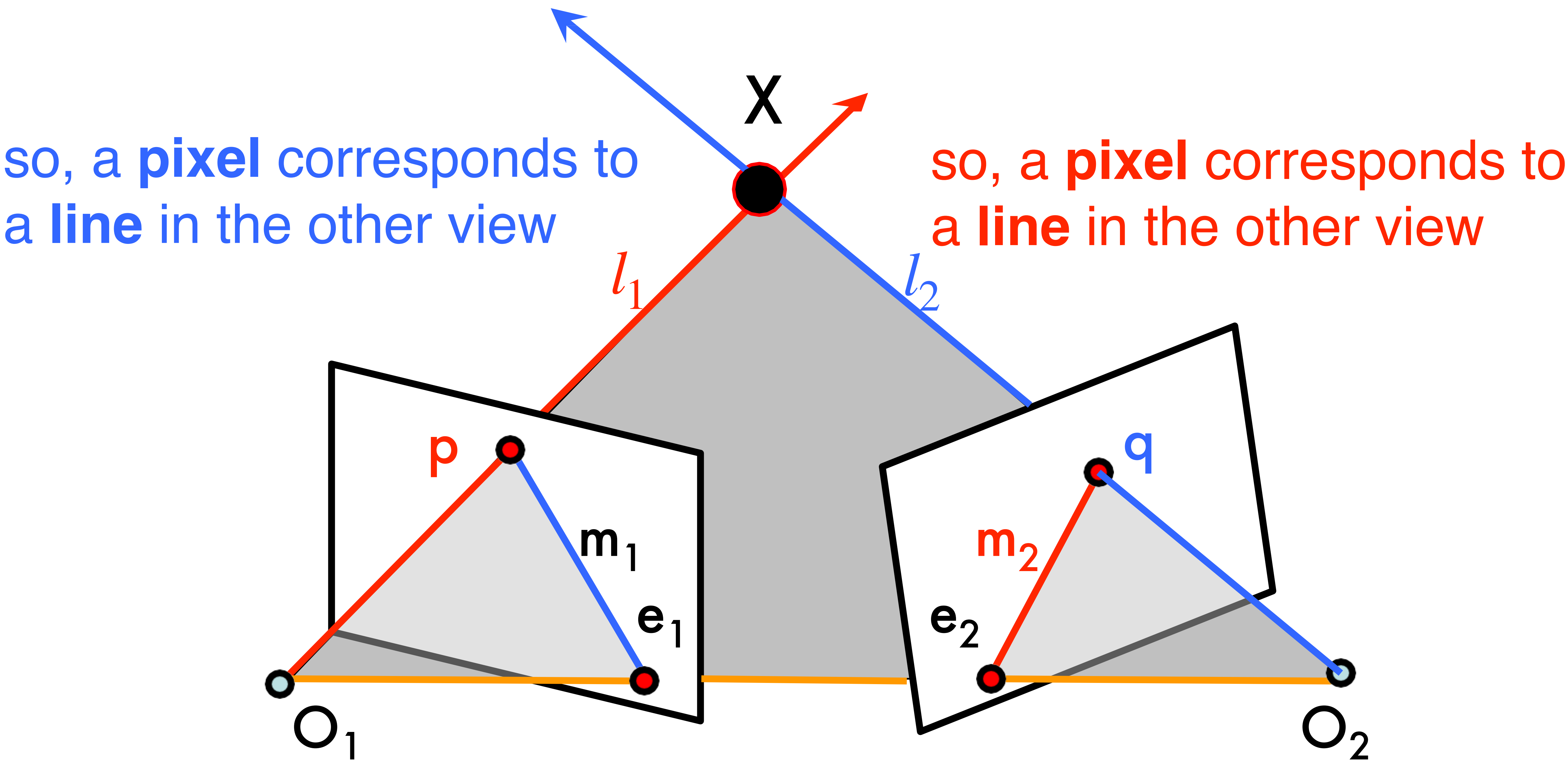
Epipolar Geometry



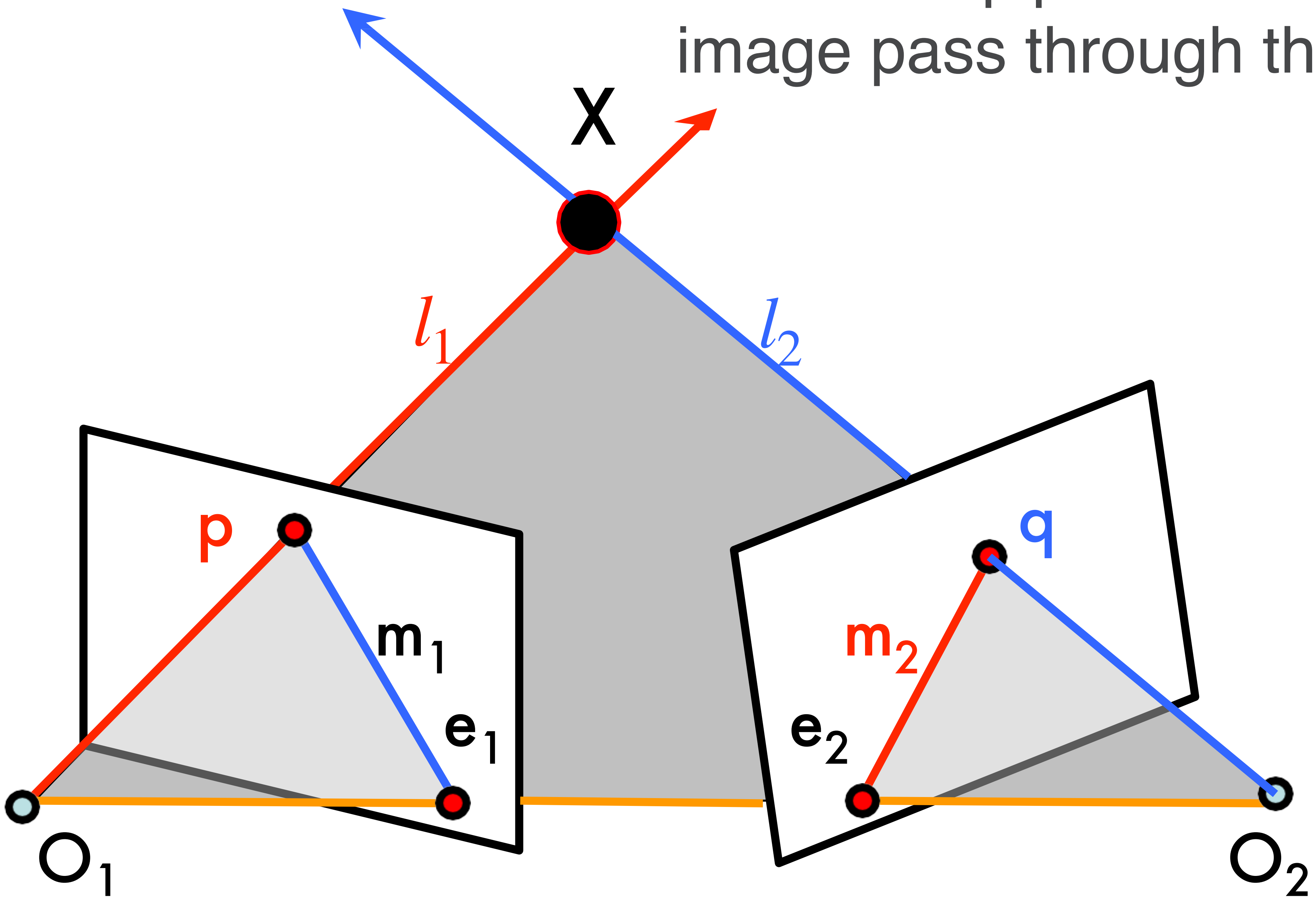
- Baselines
- Epipolar plane
- Epipolar line
- Epipoles: e_1 , e_2

  = intersections of baseline with image planes

  = projections of the other camera center

Epipolar Geometry

All of the epipolar lines in an image pass through the epipole.



- Baselines
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Agenda

- Review: Epipolar Geometry
- **Essential Matrix**
- Fundamental matrix
- Estimating F

Cross Product as Matrix Multiplication

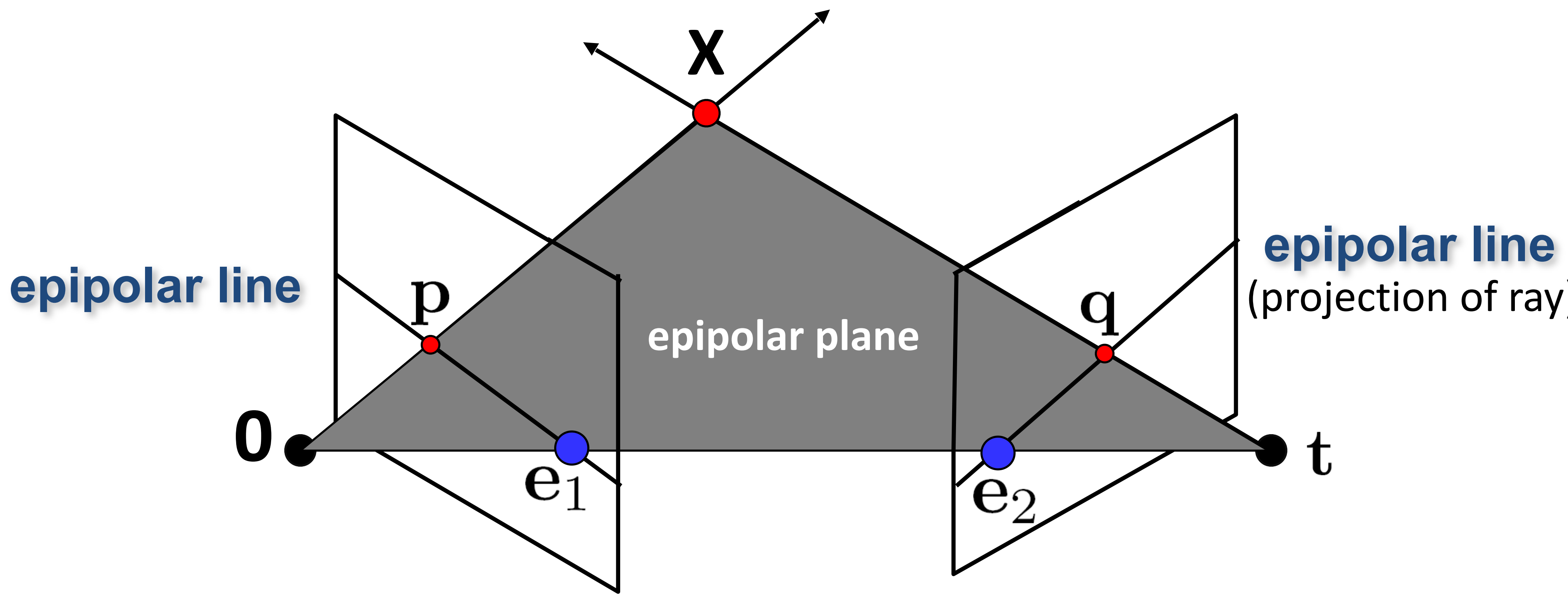
$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

$$[a_\times] = -[a_\times]^T$$

"skew-symmetric matrix"

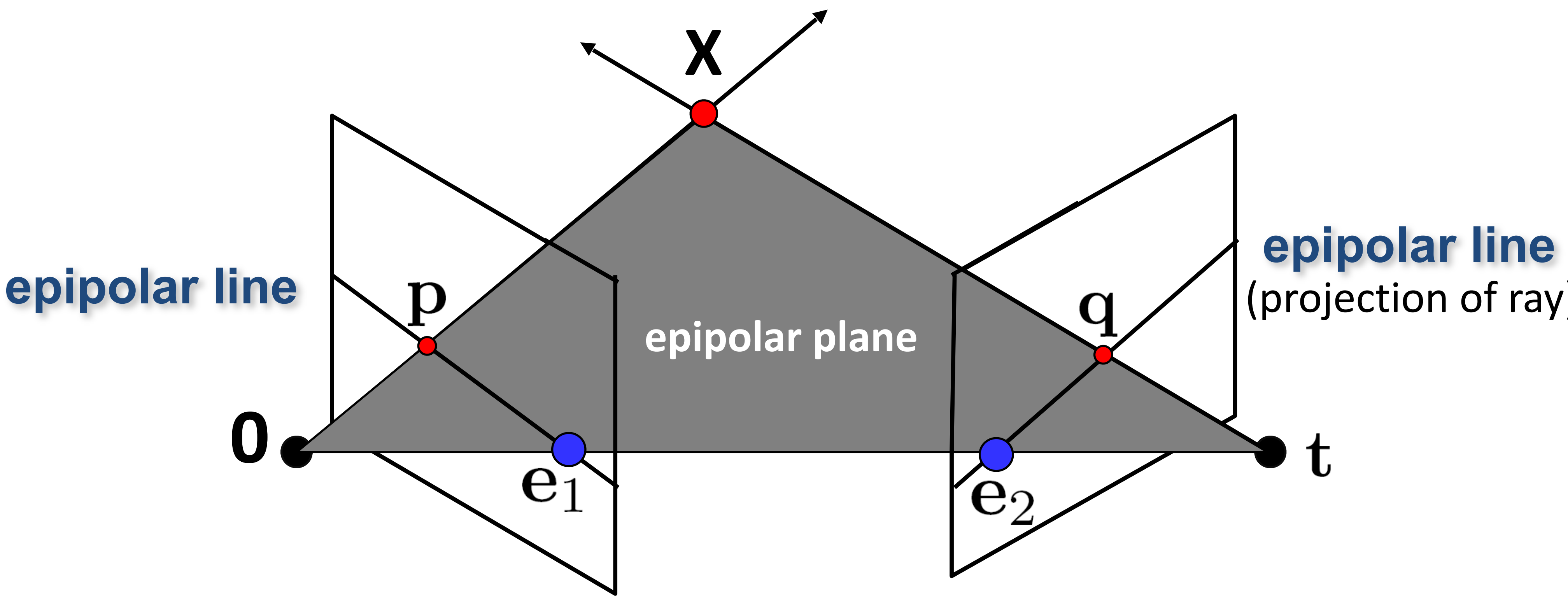
rank 2

Essential Matrix



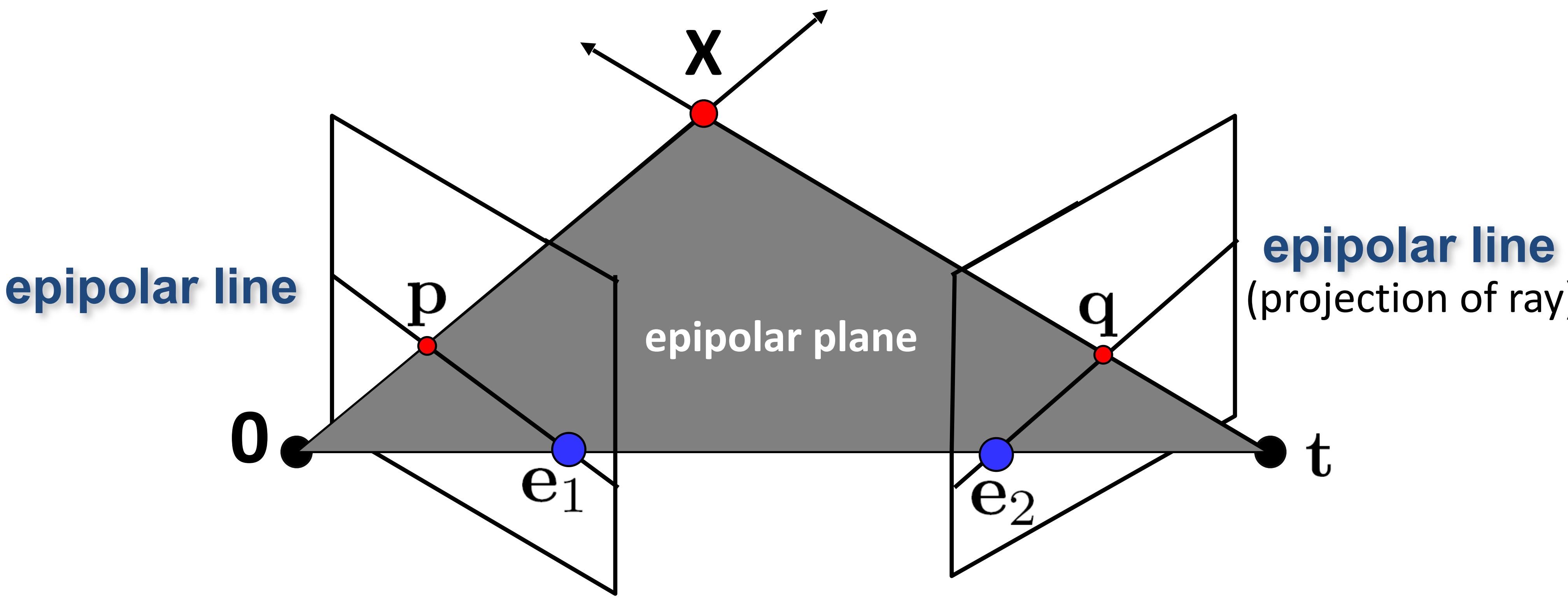
- Assume p and q in $\mathbb{R}^3$ are two points on the (virtual) image plane of two cameras
- Denoted by the pinhole frame coordinate in the corresponding cameras

Essential Matrix



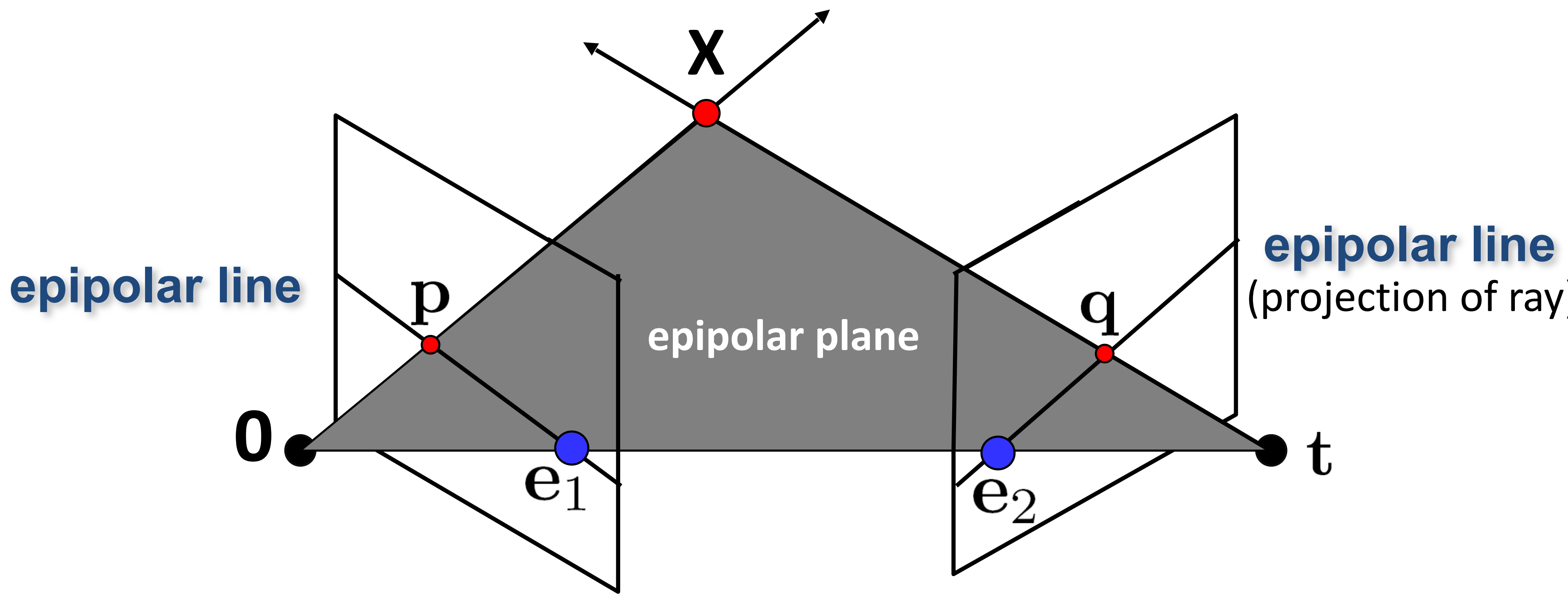
- In camera 1 pinhole frame, 3D point **X** is given by $\mathbf{X}_1 = \lambda_1 \mathbf{p}$

Essential Matrix



- In camera 1 pinhole frame, 3D point **X** is given by $\mathbf{X}_1 = \lambda_1 \mathbf{p}$
- In camera 2 pinhole frame, 3D point **X** is given by $\mathbf{X}_2 = \lambda_2 \mathbf{q}$

Essential Matrix



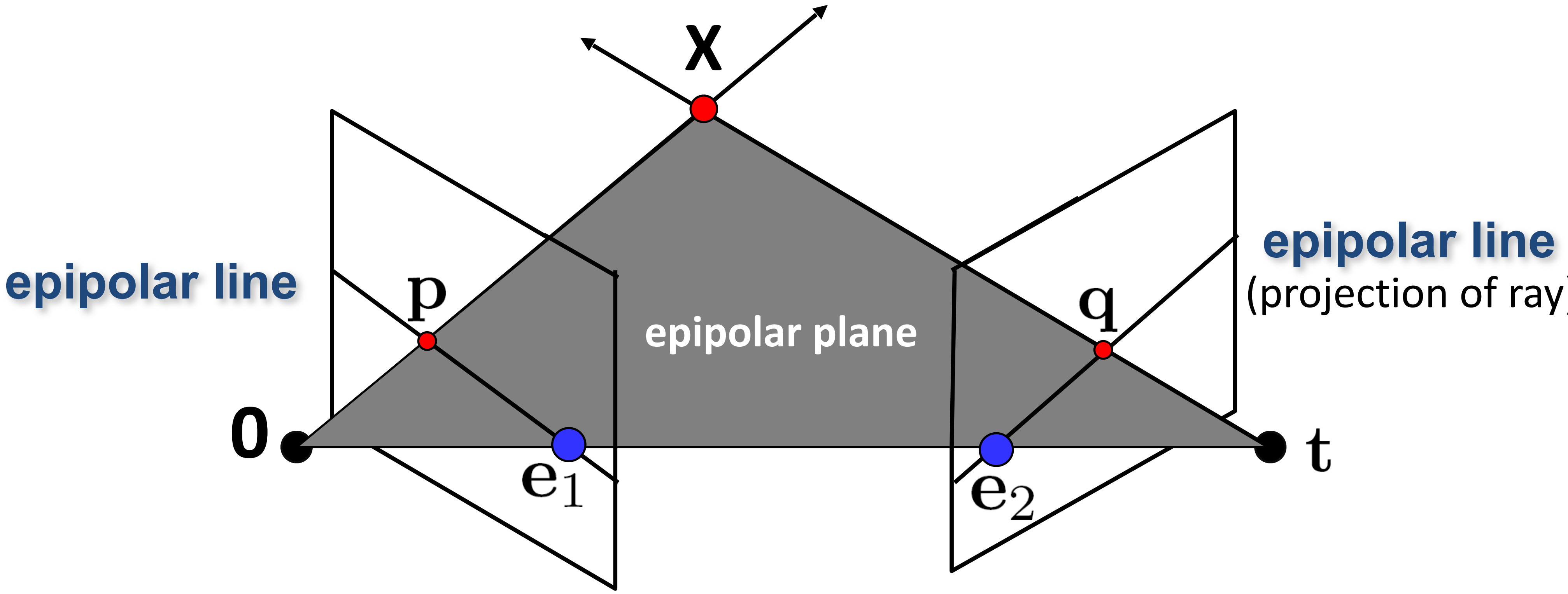
- In camera 1 pinhole frame, 3D point **X** is given by $X^1 = \lambda_1 p^1$.
- In camera 2 pinhole frame, 3D point **X** is given by $X^2 = \lambda_2 q^2$.
- Assume [**R**, **t**] changes the coordinate in camera 2 to the coordinate in camera 1

$$X^1 = RX^2 + t$$

(change of coordinate system)

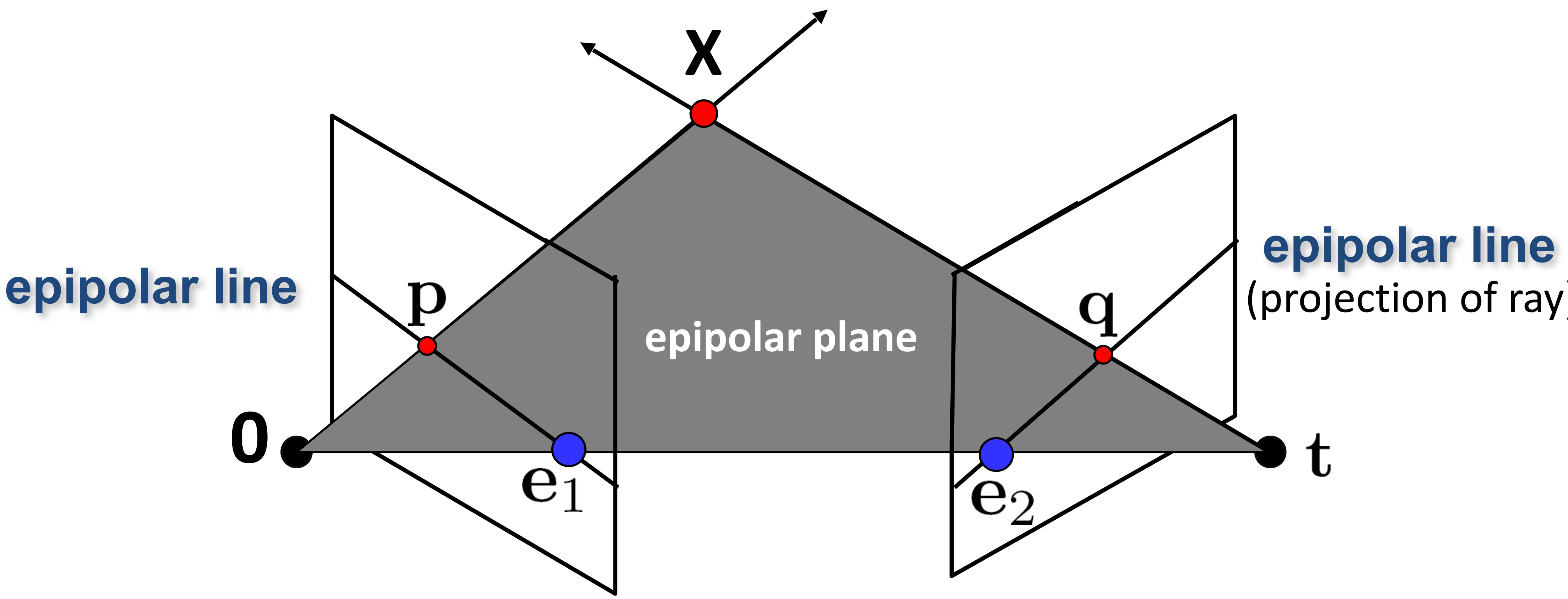
$$\lambda_1 p^1 = \lambda_2 R q^2 + t$$

Essential Matrix



- We have: $\lambda_1 p^1 = \lambda_2 R q^2 + t$

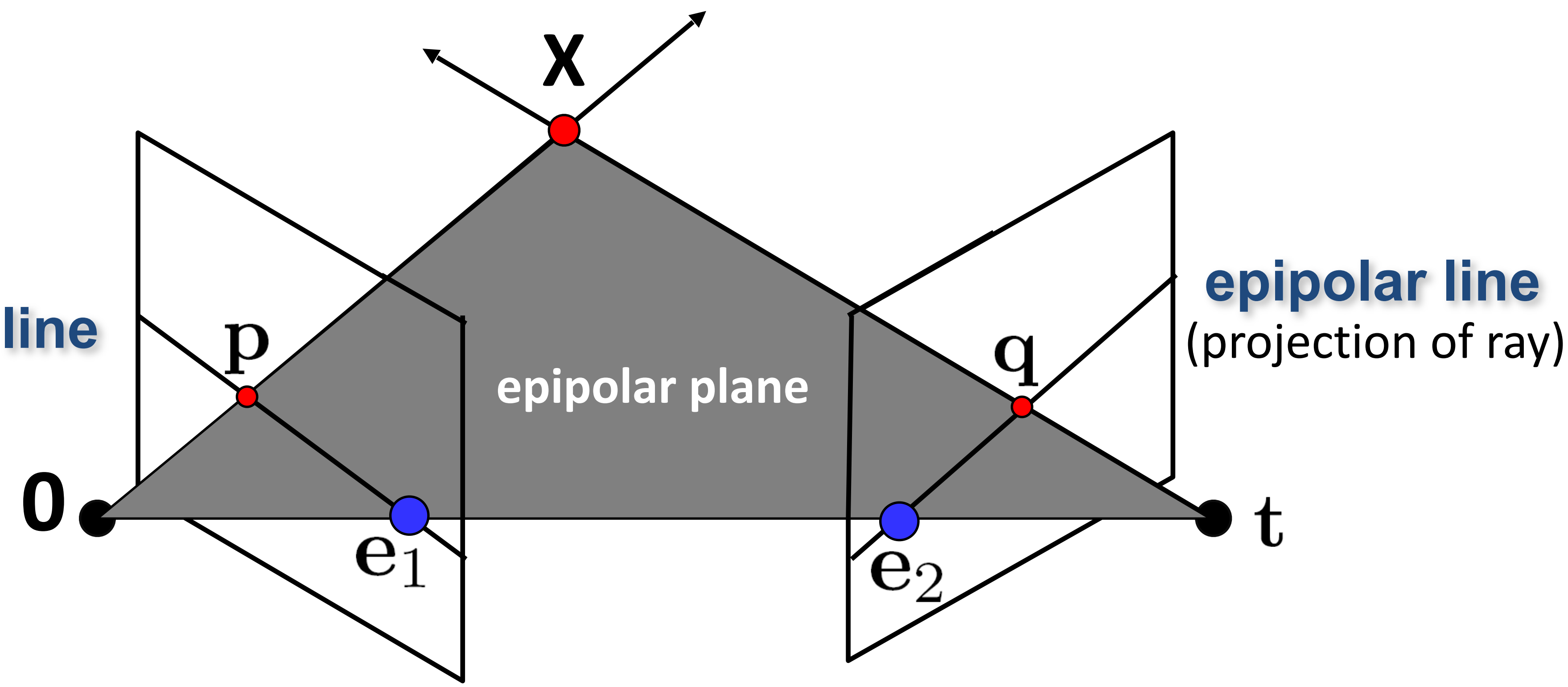
Essential Matrix



- We have: $\lambda_1 p^1 = \lambda_2 R q^2 + t$
- Take cross-product with respect to **t**:

$$\lambda_1 [t]_\times p^1 = \lambda_2 [t]_\times (R q^2 + t)$$

Essential Matrix



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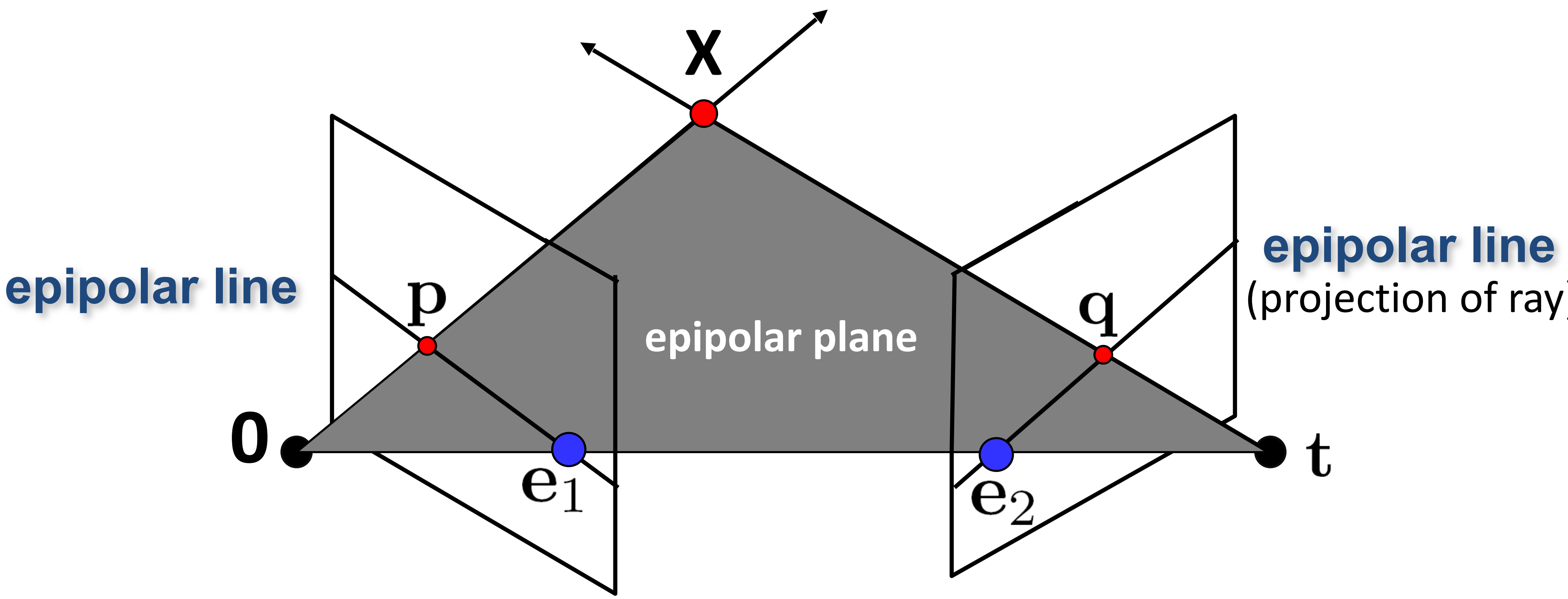


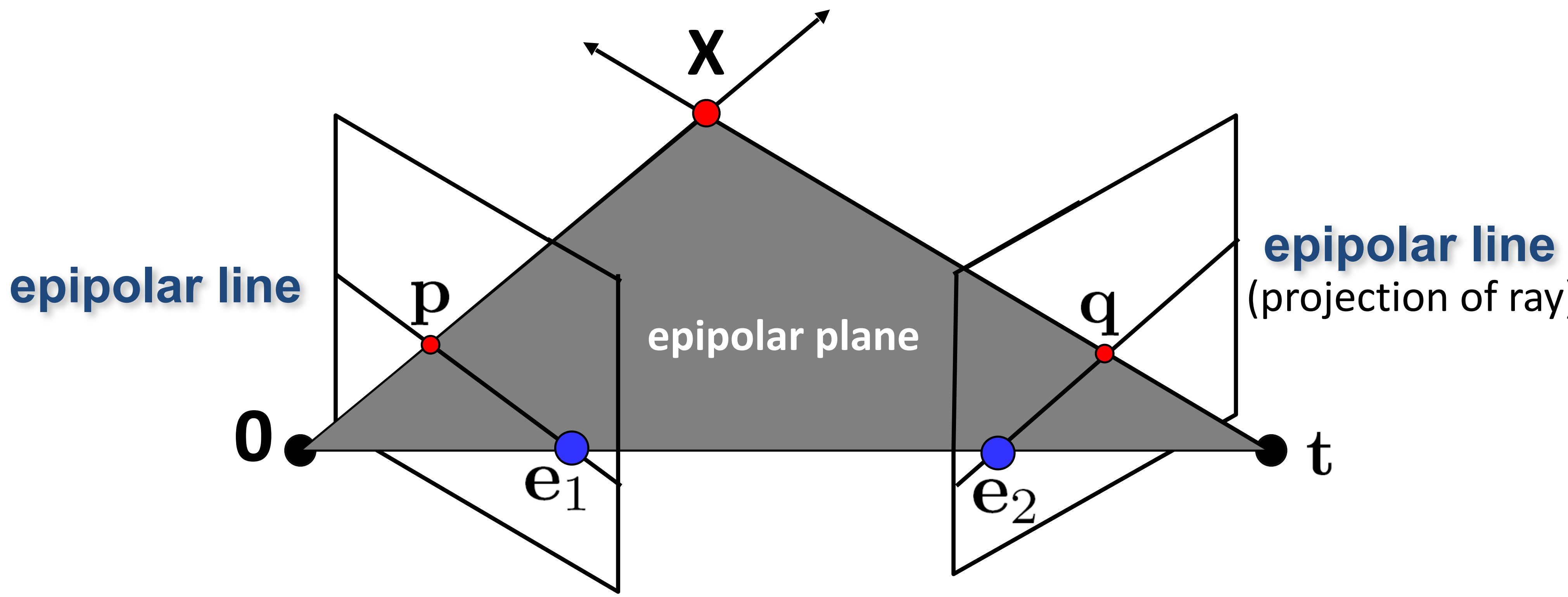
- We have: $\lambda_1 p^1 = \lambda_2 R q^2 + t$
- Take cross-product with respect to **t**:

$$\lambda_1 [t]_\times p^1 = \lambda_2 [t]_\times (R q^2 + t)$$

- Take dot-product with respect to p^1:

$$0 = \lambda_2 (p^1)^T [t]_\times R q^2$$

Essential Matrix



- We have: $(p^1)^T [t]_\times R q^2 = 0$
- Define: $\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$ ← Essential matrix

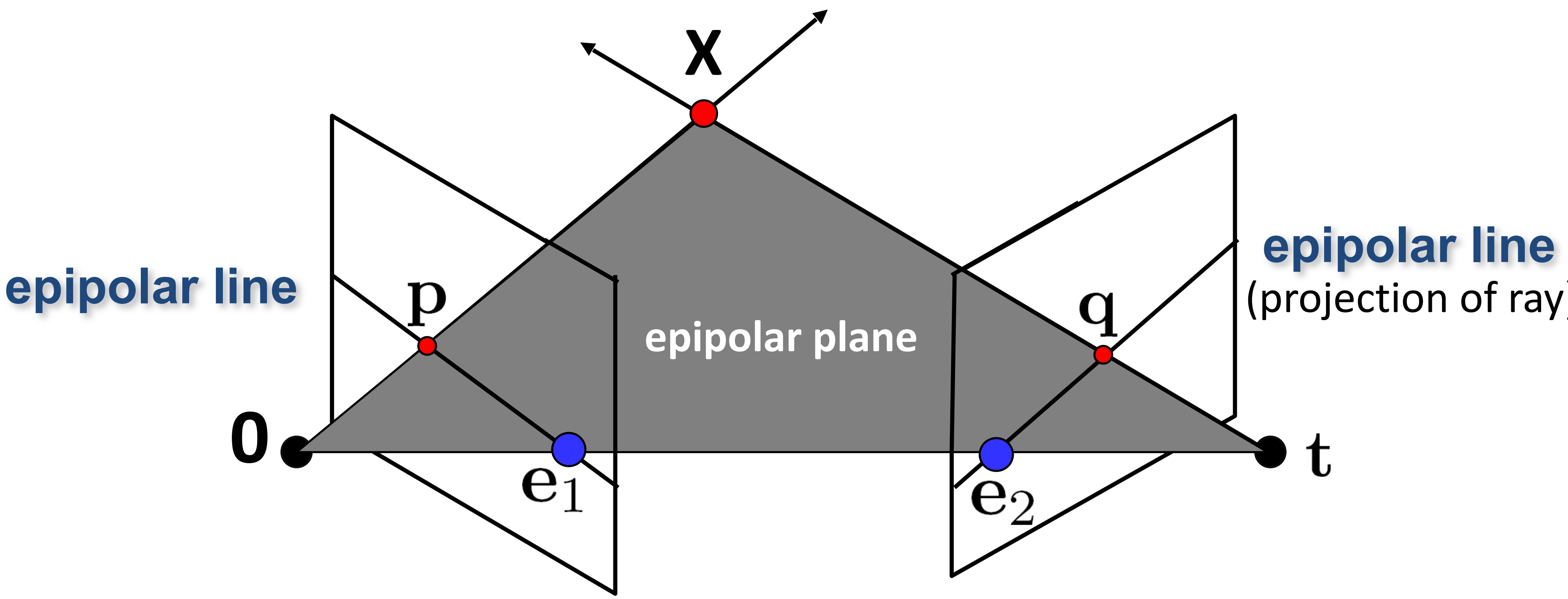
  rank(E)=2
- Then, we have:

$$(p^1)^T E q^2 = 0 \xrightarrow[\text{superscript}]{\text{omit}} p^T E q = 0$$

Agenda

- Review: Epipolar Geometry
- Essential Matrix
- **Fundamental matrix**
- Estimating F

Fundamental Matrix



- Consider intrinsic camera matrices
- Then, **p** and **q** are in the pinhole frame and pixel counterparts are:

$$\mathbf{p}' = \mathbf{K}_1\mathbf{p} \qquad \mathbf{q}' = \mathbf{K}_2\mathbf{q}$$

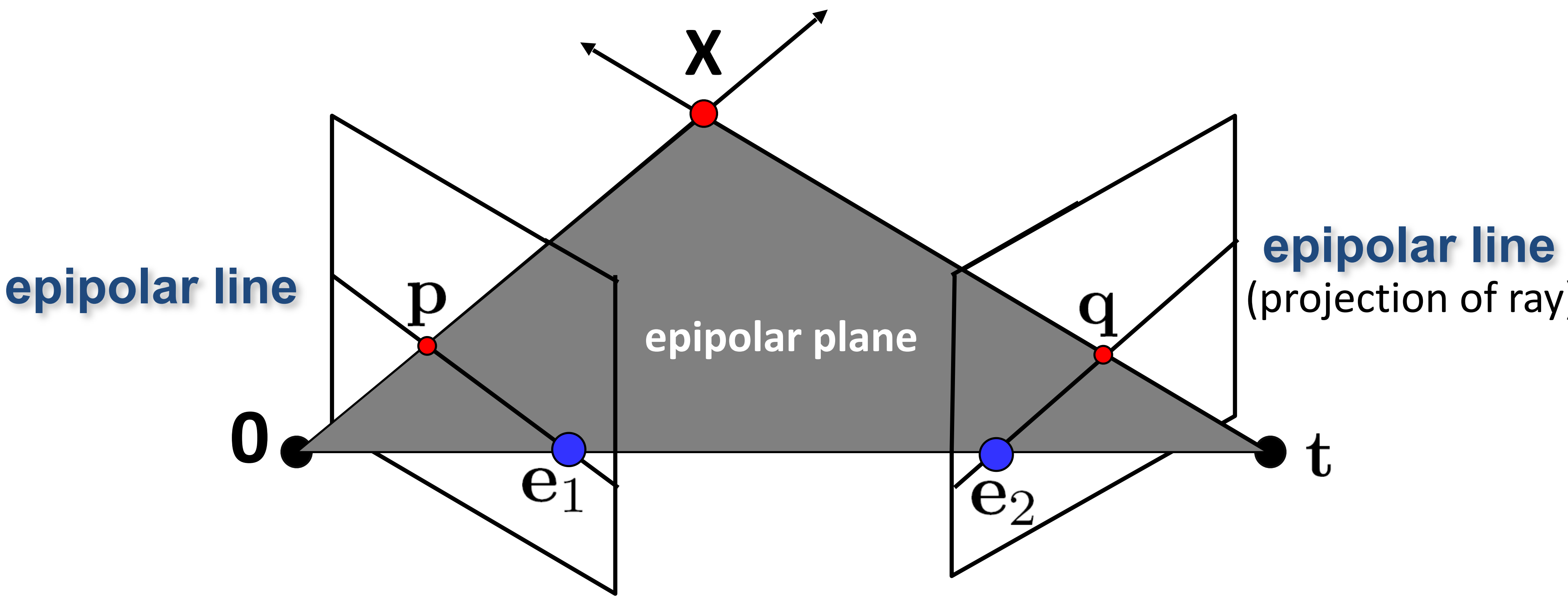
- Recall essential matrix constraint:

$$\boxed{p^T E q = 0}$$

- Substituting, we have:

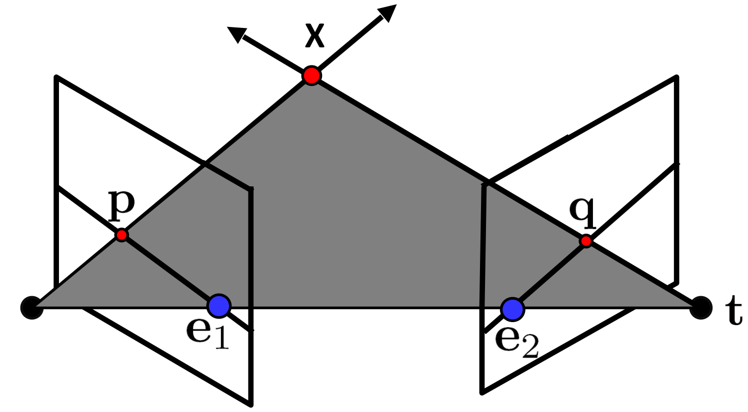
$$(K_1^{-1}p')^T E (K_2^{-1}q') = 0$$

Fundamental Matrix



- Essential matrix constraint in pixel space: $(K_1^{-1}p')^T E(K_2^{-1}q') = 0$.
- Rearranging: $p'^T K_1^{-T} E K_2^{-1} q' = 0$
- Define: $F = K_1^{-T} E K_2^{-1}$ ← Fundamental matrix   rank(F)=2
- Then, we have: $p'^T F q' = 0$

Fundamental Matrix



$$p'^T F q' = 0$$

$$[x_p, y_p, z_p] \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix} = 0$$

$\cdot / z_p$ $\quad$ $\cdot / z_q$

$$[x_p/z_p, y_p/z_p, 1] \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x_q/z_q \\ y_q/z_q \\ 1 \end{bmatrix} = 0$$

pixel coordinates

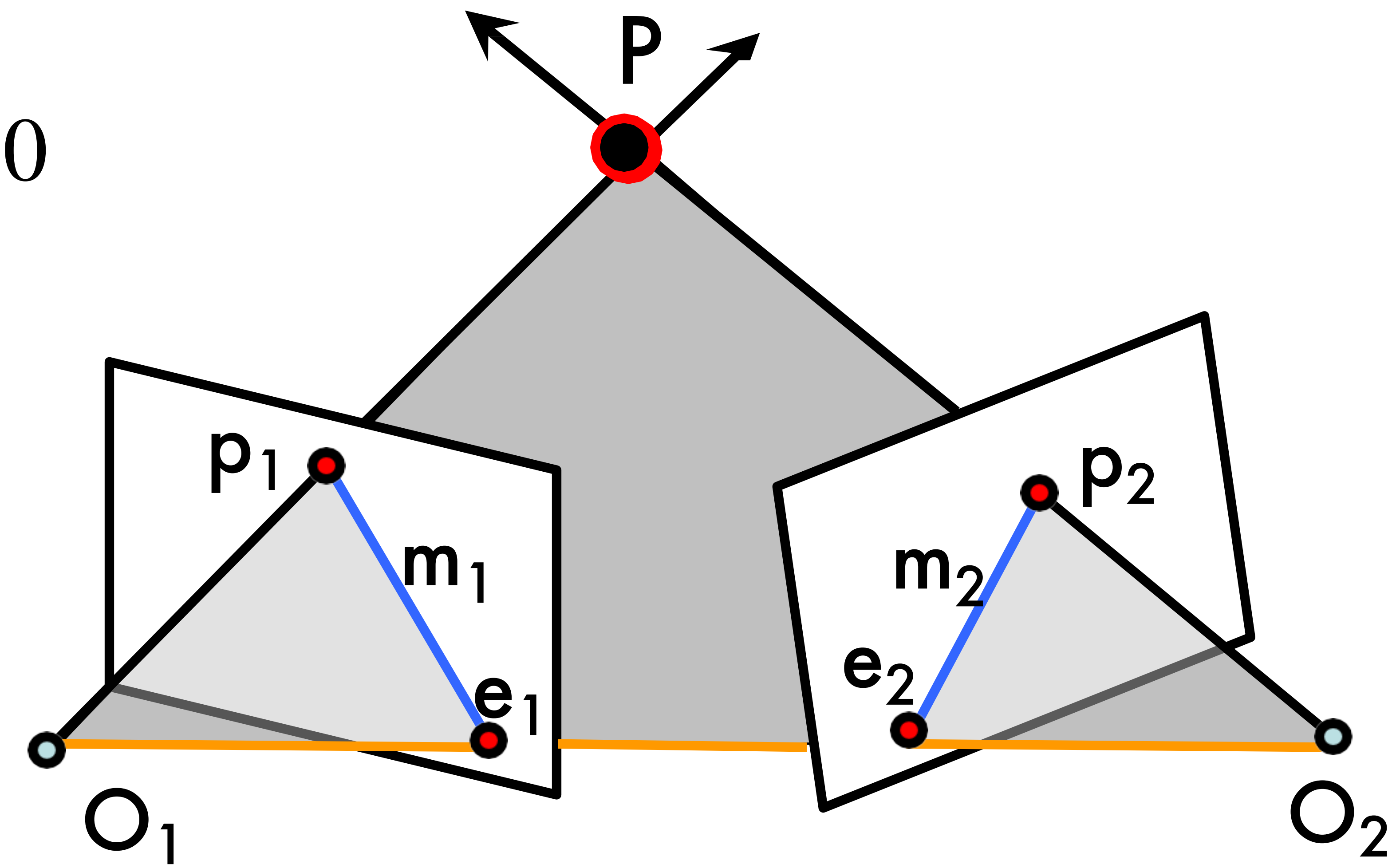
Epipolar Constraint

$$p_1^T \cdot Fp_2 = 0$$

- $w_1 = Fp_2$ defines an equation $w_1^T p_1 = 0$
- Note that, p_1 is the corresponding point of p_2 by the derivation of F
- So, $w_1 = Fp_2$ defines the epipolar line of p_2

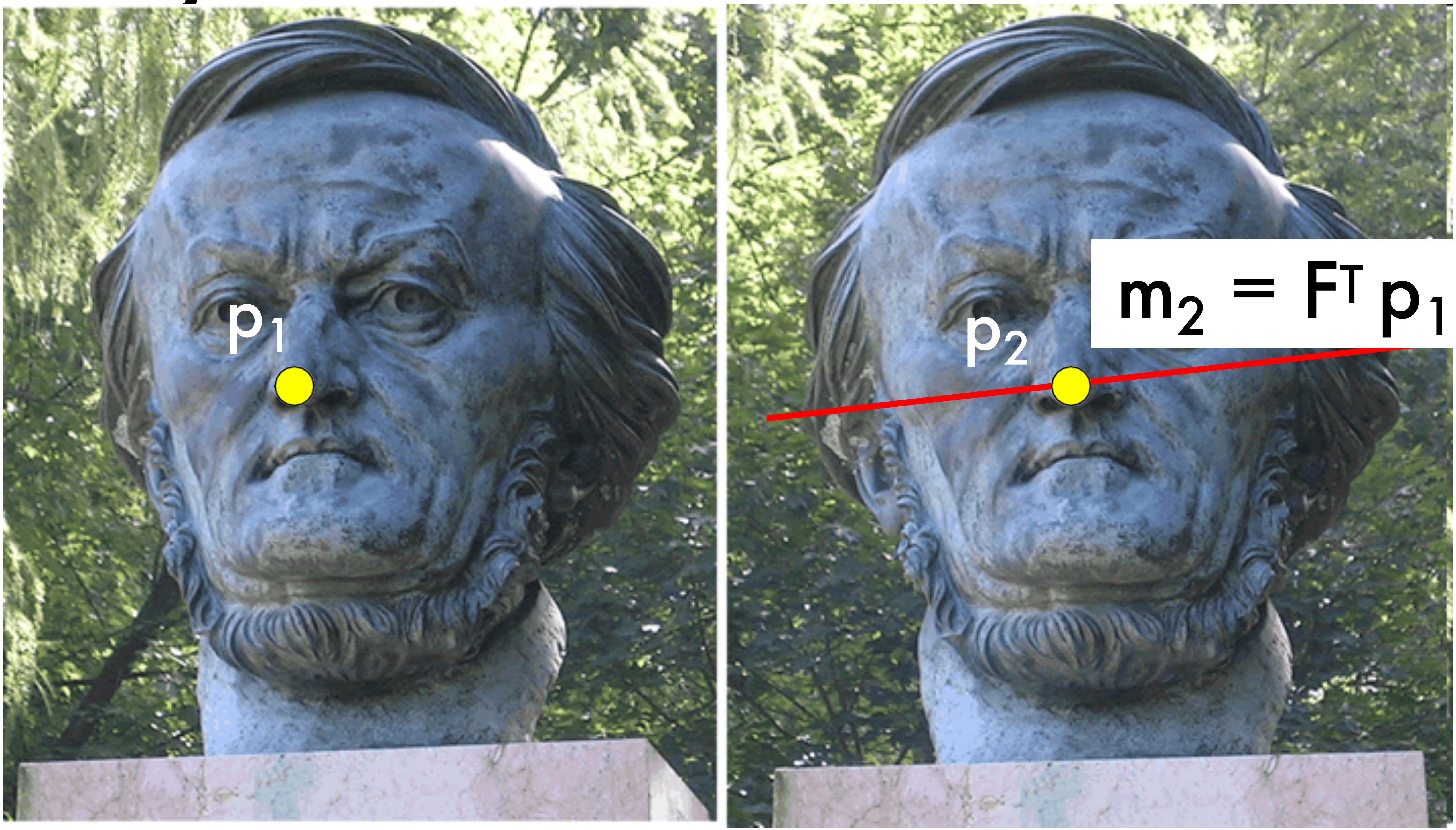
Epipolar Constraint

$$p_1^T \cdot F p_2 = 0$$



- w_1 = F p_2 defines an equation $w_1^T p_1 = 0$, the epipolar line m_1 of p_2
- $w_2 = F^T p_1$ defines an equation $w_2^T p_2 = 0$, the epipolar line m_2 of p_1
- F is singular (rank two)
- F e_2 = 0 and $F^T e_1 = 0$

Why F is useful?



- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second image

Why F is useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- MORE IMPORTANTLY: F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

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- Review: Epipolar Geometry
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- **Estimating F**

Estimating F

Suppose we have a pair of corresponding points:

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

[Eq. 13] $$\mathrm{p}^{\mathrm{T}} \, \mathrm{F} \, \mathrm{p}' = 0 \quad \Rightarrow$$

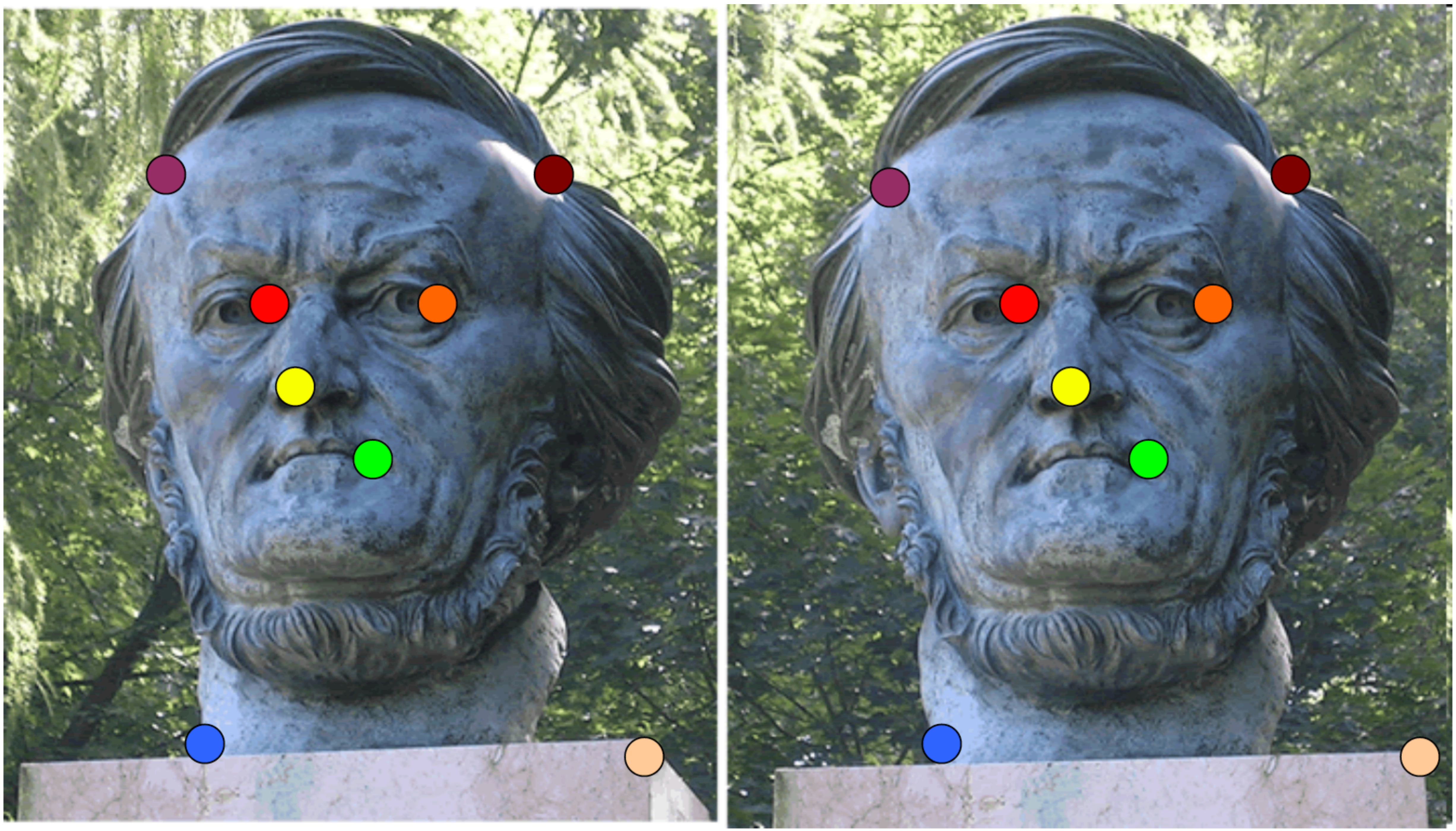
$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad (uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Estimating F



Estimating F

$$\left(u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v_i', 1\right)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Estimating F

$$\mathbf{W}\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad \mathbf{f}$$

[Eqs. 15]

- Homogeneous system $\mathbf{W}\,\mathbf{f} = 0$
- Rank 8 ⟶ A non-zero solution exists (unique)
- If N>8 ⟶ Lsq. solution by SVD! ⟶ $\hat{\mathrm{F}}$

$\|\mathbf{f}\| = 1$

$\hat{F}$ satisfies: $p^T \hat{F} p' = 0$

and estimated $\hat{F}$ may have full rank ($\det(\hat{F}) \neq 0$)

But remember: fundamental matrix is Rank2

Find F that minimizes $\left\| F - \hat{F} \right\|$ Frobenius norm (*)

Subject to rank(F)=2

SVD (again!) can be used to solve this problem

(*) Sq. root of the sum of squares of all entries

Find F that minimizes $\left\| \mathrm{F} - \hat{\mathrm{F}} \right\|$

Frobenius norm (*)

Subject to det(F)=0

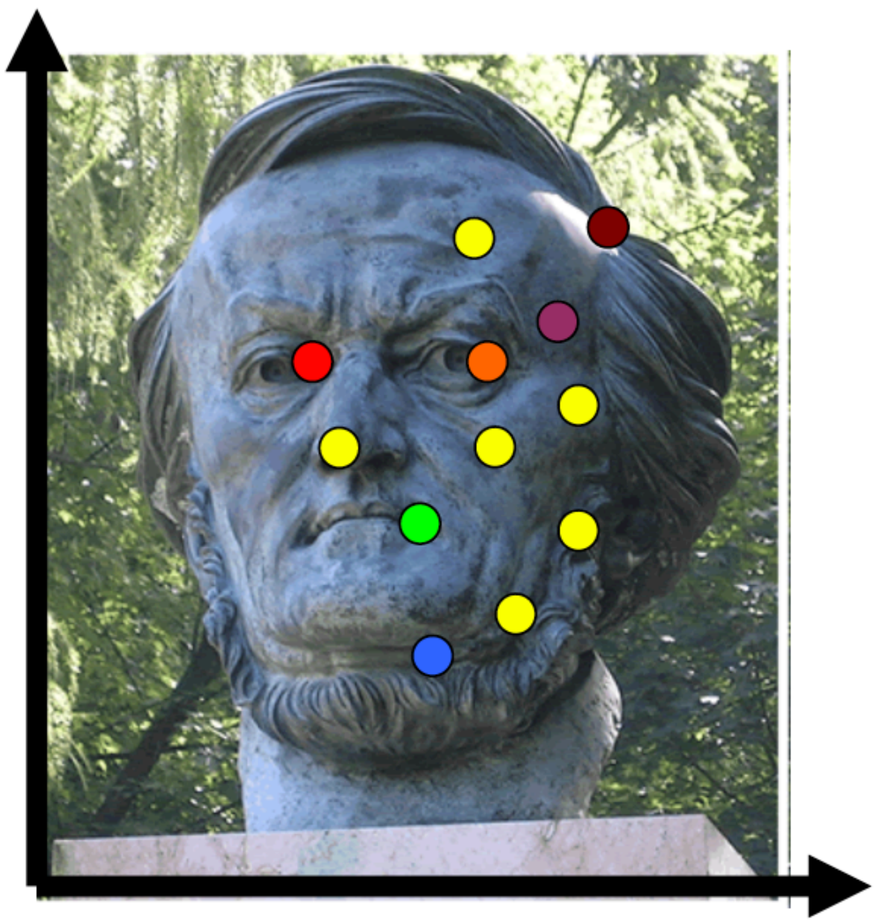
$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, "Computation of F"

Problems with the 8-Point Algorithm



$$Wf = 0, \; \|f\| = 1$$

↓ Least-square

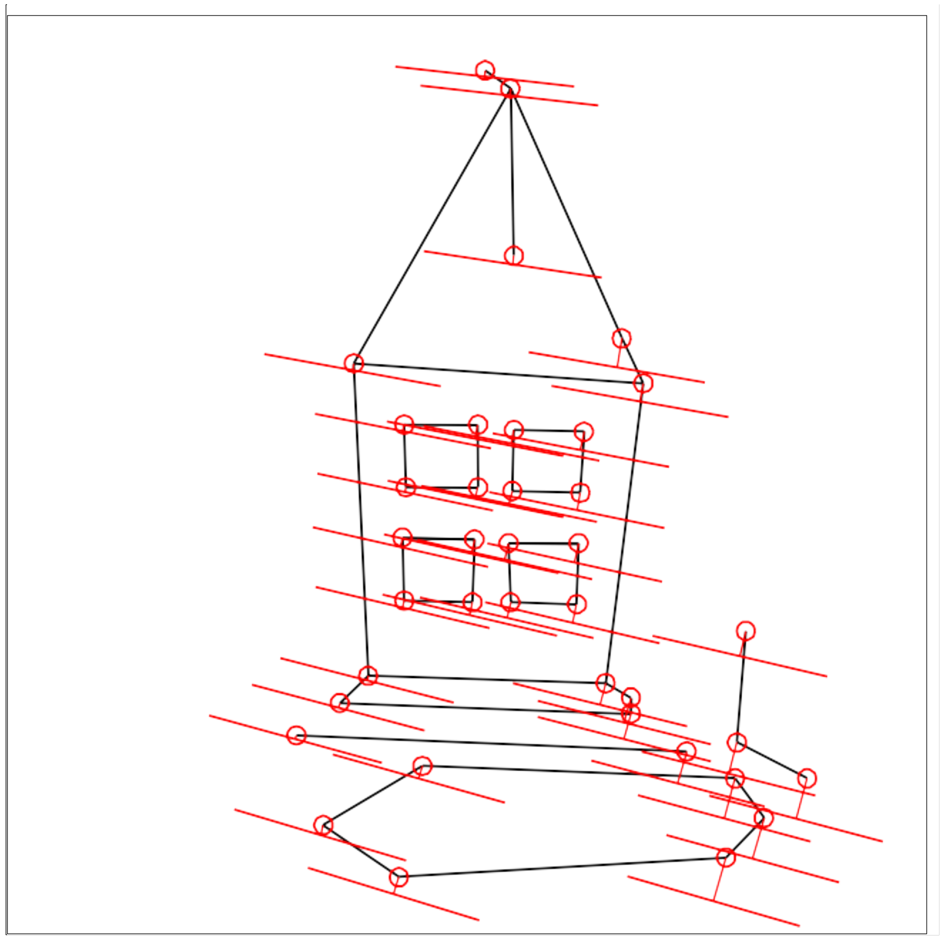
$$\underset{f}{\textbf{minimize}} \quad \|Wf\|^2$$
$$s.t. \quad \|f\| = 1$$

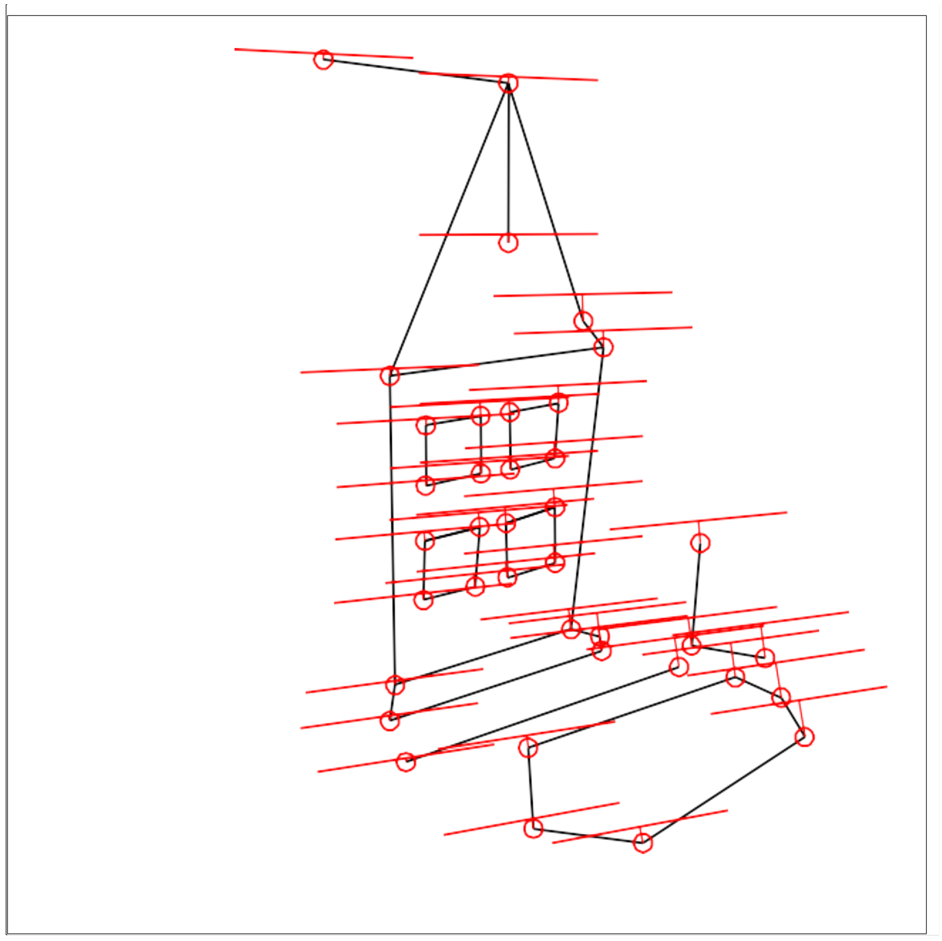
↓

$$\underset{f}{\textbf{minimize}} \quad f^T W^T W f$$
$$s.t. \quad f^T f = 1$$

↓

Do you remember how to solve the problem?
Hint: Check your HW1 (by the SVD of W)





Mean errors:
10.0pixel
9.1pixel