CSE 152: Computer Vision

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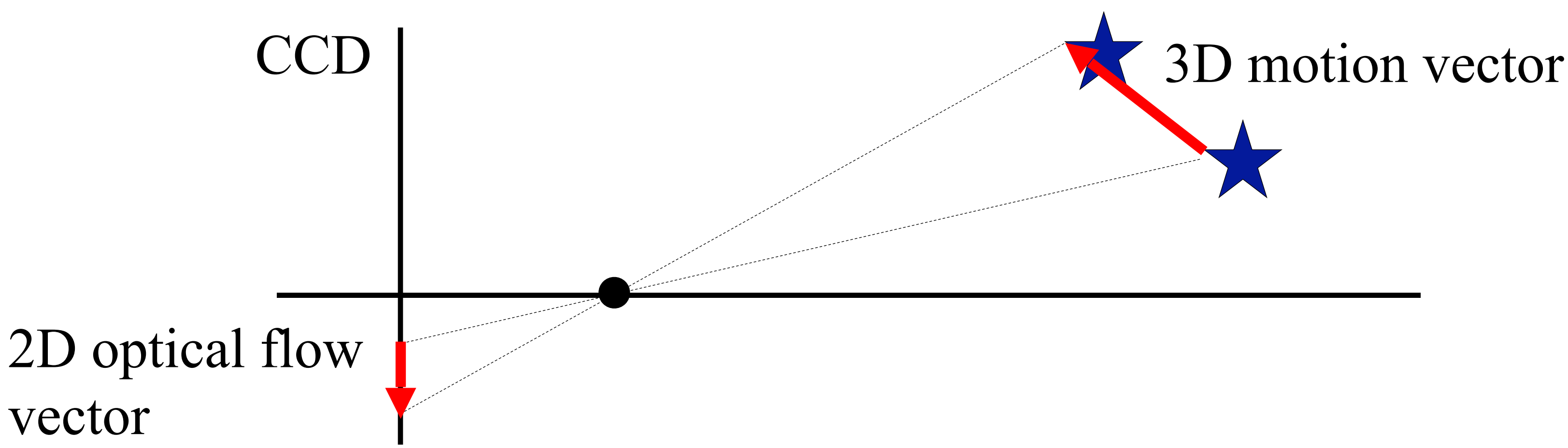
Lecture 18: Lucas-Kanade Algorithm

Motion Field & Optical Flow Field

- Motion Field = Real world 3D motion
- Optical Flow Field = Projection of the motion field onto the 2d image

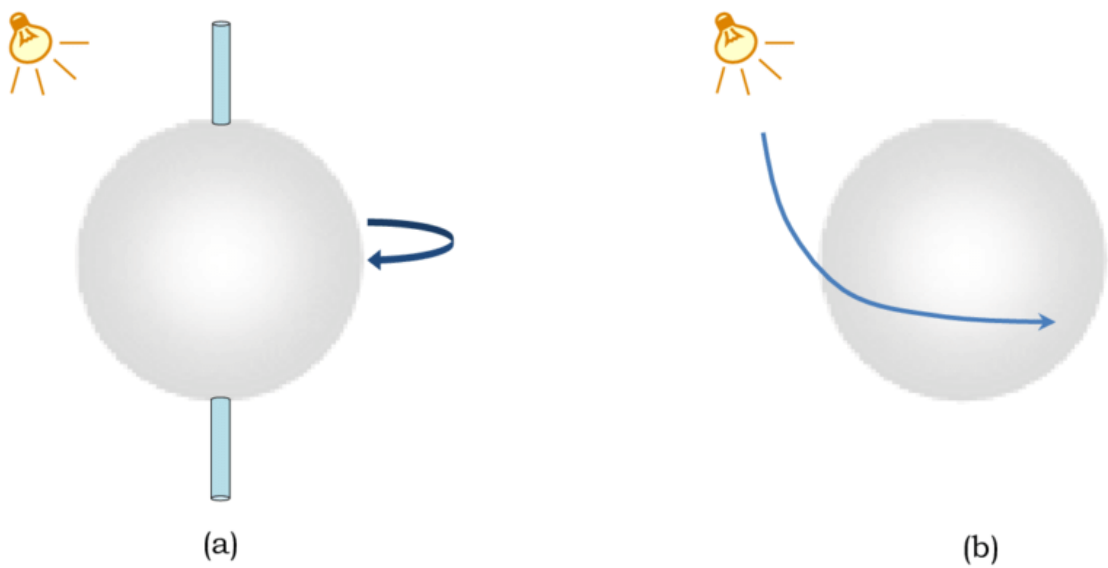


$$\vec{u} = (u, v)$$

Slide adapted from Savarese.

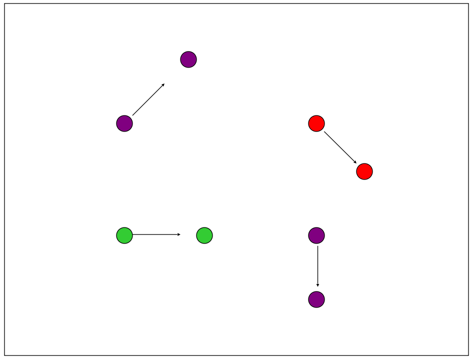
Apparent Motion

- Optical flow differs from actual motion field:
  - (a) intensity remains constant, so that no motion is perceived;
  - (b) no object motion exists, however moving light source produces shading changes.

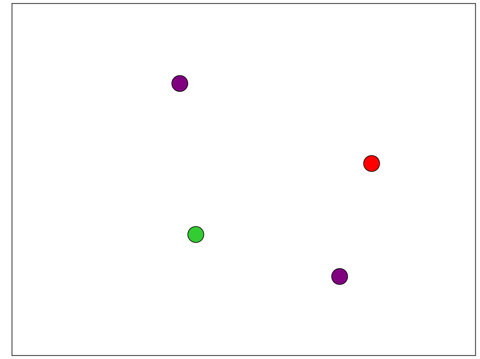


Source: Silvio Savarese

Estimating Optical Flow



$I(x,y,t-1)$

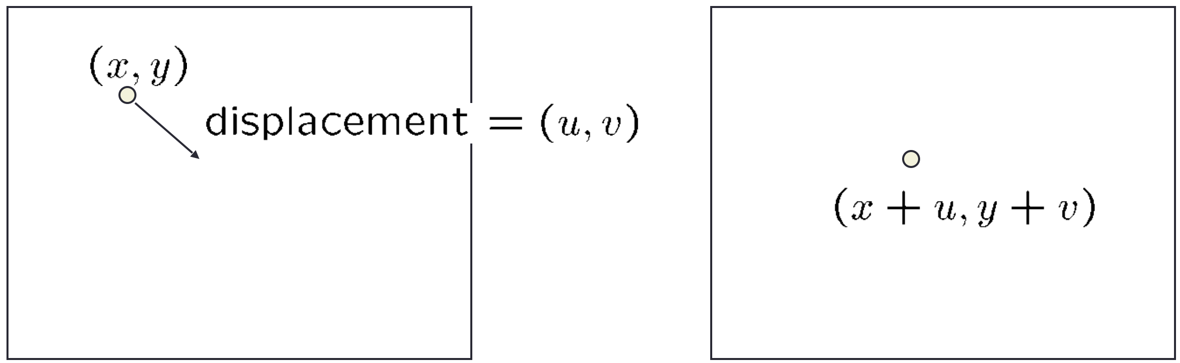


$I(x,y,t)$

- Given two subsequent frames, estimate the apparent motion field u(x,y), v(x,y) between them
- **Key assumptions**
  - Brightness constancy: projection of the same point looks the same in every frame
  - Small motion: points do not move very far
  - Spatial coherence: points move like their neighbors

Source: Silvio Savarese

Optical Flow Constraints (grayscale images)



$$I(x, y, t) \qquad\qquad I(x, y, t+1)$$

- Let's look at these constraints more closely
  - Brightness constancy constraint (equation)

$$I(x, y, t) = I(x+u, y+v, t+1)$$

  - Small motion: (u and v are less than 1 pixel, or smoothly varying)
    Taylor series expansion of I:

$$I(x+u, y+v, t+1) = I(x, y, t) + \frac{\partial I}{\partial x}u + \frac{\partial I}{\partial y}v + \frac{\partial I}{\partial t} + o(1)$$

Brightness Constancy Constraint Equation

$$I(x+u, y+v, t+1) = I(x, y, t) + \frac{\partial I}{\partial x}u + \frac{\partial I}{\partial y}v + \frac{\partial I}{\partial t} + o(1)$$

$$\boxed{I_x\, u + I_y\, v + I_t = 0}$$

The Brightness Constancy Constraint

Can we use this equation to recover image motion (u,v) at each pixel?

$$0 = I_t + \nabla I \cdot < u, v >$$

- How many equations and unknowns per pixel?
  - One equation (this is a scalar equation!), two unknowns (u,v)

Source: Silvio Savarese

The Brightness Constancy Constraint

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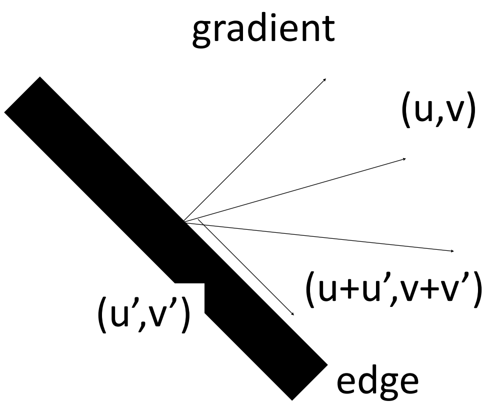
The component of the flow perpendicular to the gradient (i.e., parallel to the edge) cannot be measured

If (u, v) satisfies the equation,
so does (u+u', v+v') if

$$\nabla I \cdot \begin{bmatrix} u' & v' \end{bmatrix}^T = 0$$

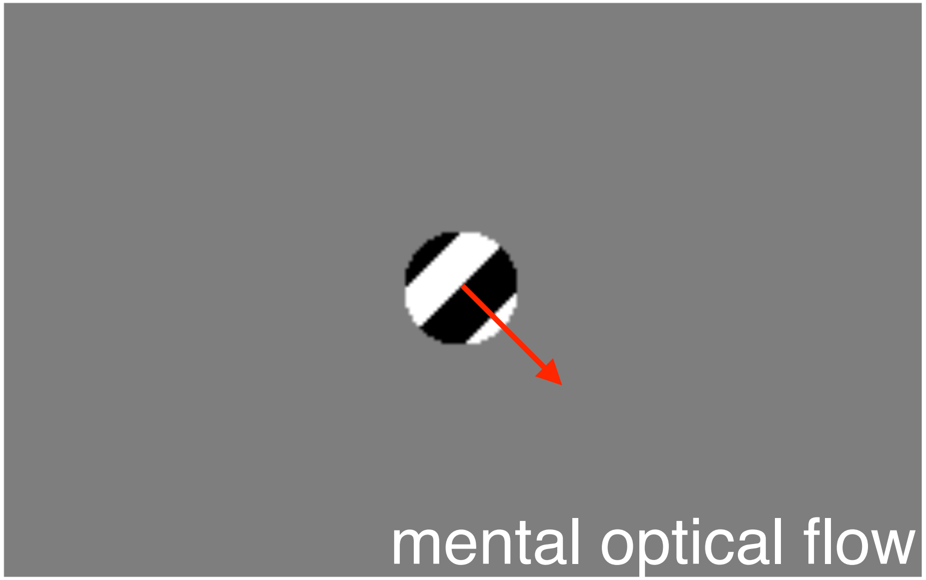


Source: Silvio Savarese

The Barber Pole Illusion

http://en.wikipedia.org/wiki/Barberpole_illusion

The Barber Pole Illusion



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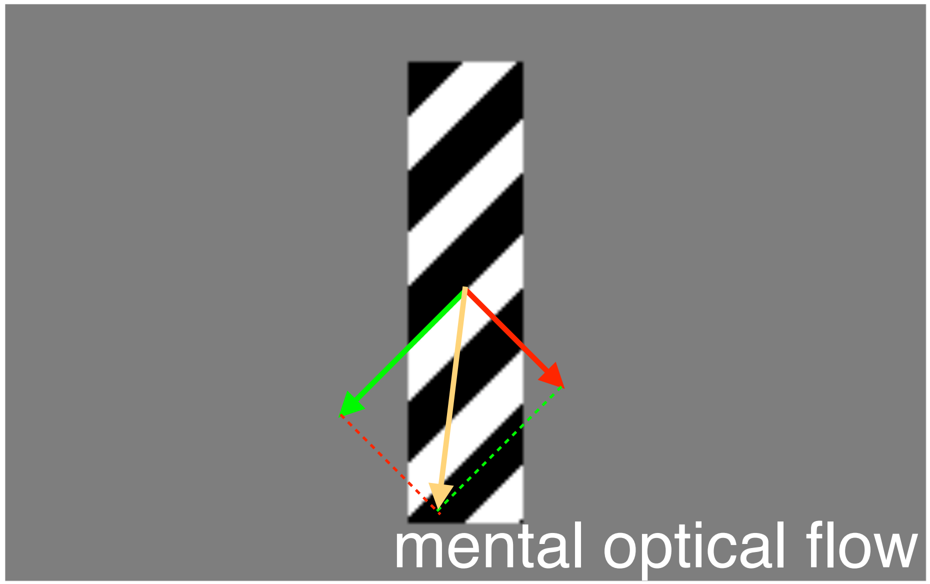

http://en.wikipedia.org/wiki/Barberpole_illusion

Solving the Ambiguity…

B. Lucas and T. Kanade. An iterative image registration technique with an application to stereo vision. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pp. 674–679, 1981.

- How to get more equations for a pixel?
- **Spatial coherence constraint**

  Assume the pixel's neighbors have the same (u,v)

  If we use a 5x5 window, that gives us 25 equations per pixel

$$0 = I_t(\mathbf{p_i}) + \nabla I(\mathbf{p_i}) \cdot [u \;\; v]$$

$$\begin{bmatrix} I_x(\mathbf{p_1}) & I_y(\mathbf{p_1}) \\ I_x(\mathbf{p_2}) & I_y(\mathbf{p_2}) \\ \vdots & \vdots \\ I_x(\mathbf{p_{25}}) & I_y(\mathbf{p_{25}}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\mathbf{p_1}) \\ I_t(\mathbf{p_2}) \\ \vdots \\ I_t(\mathbf{p_{25}}) \end{bmatrix}$$

Solving the Ambiguity…

- Least squares problem:

$$\begin{bmatrix} I_x(\mathbf{p}_1) & I_y(\mathbf{p}_1) \\ I_x(\mathbf{p}_2) & I_y(\mathbf{p}_2) \\ \vdots & \vdots \\ I_x(\mathbf{p}_{25}) & I_y(\mathbf{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\mathbf{p}_1) \\ I_t(\mathbf{p}_2) \\ \vdots \\ I_t(\mathbf{p}_{25}) \end{bmatrix}$$

$$\underset{25\times 2}{A} \; \underset{2\times 1}{d} = \underset{25\times 1}{b}$$

Matching Matches Across Images

- Overconstrained linear system

$$\begin{bmatrix} I_x(\mathbf{p}_1) & I_y(\mathbf{p}_1) \\ I_x(\mathbf{p}_2) & I_y(\mathbf{p}_2) \\ \vdots & \vdots \\ I_x(\mathbf{p}_{25}) & I_y(\mathbf{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\mathbf{p}_1) \\ I_t(\mathbf{p}_2) \\ \vdots \\ I_t(\mathbf{p}_{25}) \end{bmatrix} \qquad \underset{25\times 2}{A} \; \underset{2\times 1}{d} = \underset{25\times 1}{b}$$

Least squares solution for *d* given by $(A^T A) \; d = A^T b$

Matching Matches Across Images

- Overconstrained linear system

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Least squares solution for *d* given by $(A^T A)\; d = A^T b$

$$A = \begin{bmatrix} (\nabla I(p_1))^T \\ \vdots \\ (\nabla I(p_n))^T \end{bmatrix} \Rightarrow A^T A = \sum_i \nabla I(p_i)(\nabla I(p_i))^T = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix}$$

The summations are over all pixels in the K x K window

Conditions for Solvability

Optimal (u, v) satisfies Lucas-Kanade equation

$$\begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}$$

$$A^T A \qquad\qquad\qquad\qquad A^T b$$

When is this solvable? What are good points to track?

- **A^{T}A** should be invertible
- **A^{T}A** should not be too small due to noise
  - eigenvalues λ_1 and λ_2 of **A^{T}A** should not be too small
- **A^{T}A** should be well-conditioned
  - λ_1 / λ_2 should not be too large (λ_1 = larger eigenvalue)

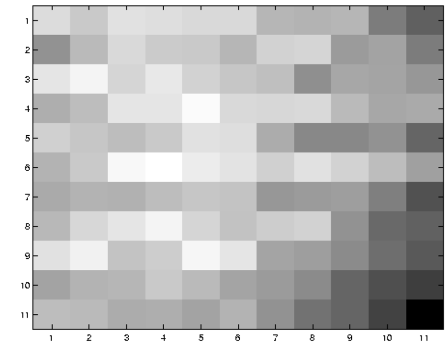
Link Linear Algebra with Pixels

Let $A = \begin{bmatrix} (\nabla I(p_1))^T \\ \vdots \\ (\nabla I(p_n))^T \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$

$\forall w \in \mathbb{R}^2$, we have $w^T(A^TA)w = \|Aw\|^2 = \sum (a_i^T w)^2$

$$\lambda_1 = \lambda_{max}(A^TA) \qquad \lambda_2 = \lambda_{min}(A^TA)$$

Low Texture Region



$$A = \begin{bmatrix} (\nabla I(p_1))^T \\ \vdots \\ (\nabla I(p_n))^T \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$$

Gradients (a_i) have small magnitude

Low Texture Region

Gradients (a_i) have small magnitude

$$A = \begin{bmatrix} (\nabla I(p_1))^T \\ \vdots \\ (\nabla I(p_n))^T \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$$

$$w^T(A^TA)w = \sum (a_i^T w)^2$$

↓

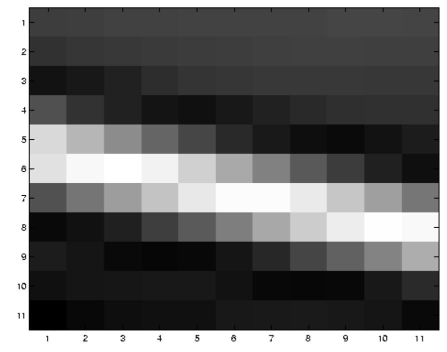
small for any w with $\|w\|^2 = 1$

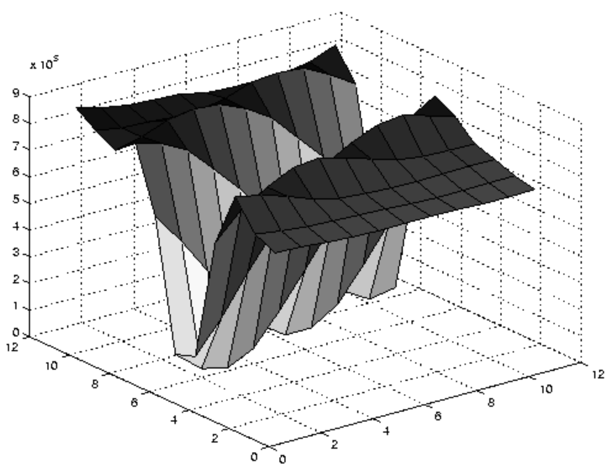
Because $\underset{\|w\|^2=1}{\text{maximize}}\ w^T(A^TA)w = \lambda_1$ (Recall 8-point algo. and HW1)

↓

λ_1 and λ_2 are both small

Edge





large gradients, all the same

Edge

Large gradients (a_i), all the same

$$A = \begin{bmatrix} (\nabla I(p_1))^T \\ \vdots \\ (\nabla I(p_n))^T \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$$

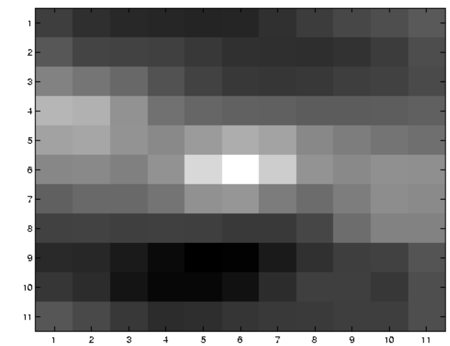
$$w^T(A^TA)w = \sum (a_i^T w)^2$$

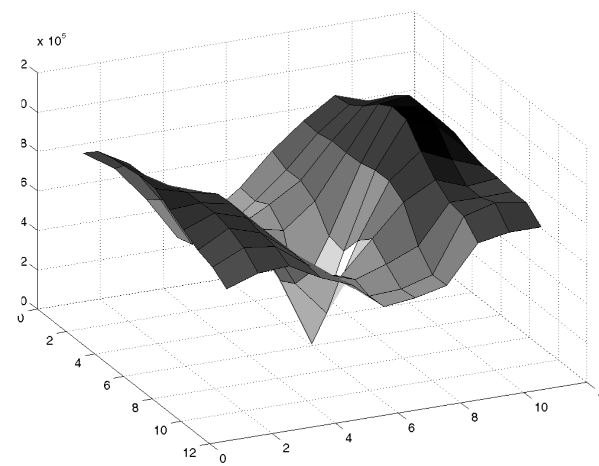
big for w parallel to gradients
small for w orthogonal to gradients

Because $\underset{\|w\|^2=1}{\text{maximize}}\ w^T(A^TA)w = \lambda_1$, λ_1 is big

Because $\underset{\|w\|^2=1}{\text{minimize}}\ w^T(A^TA)w = \lambda_2$, λ_2 is small

High Textured Region





gradients are different,
large magnitudes

High Textured Region

gradients (a_i) are different, large magnitudes

$$A = \begin{bmatrix} (\nabla I(p_1))^T \\ \vdots \\ (\nabla I(p_n))^T \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$$

$$w^T(A^TA)w = \sum (a_i^T w)^2$$

↓

for any w there are some $a_i's$ with signficant component along it

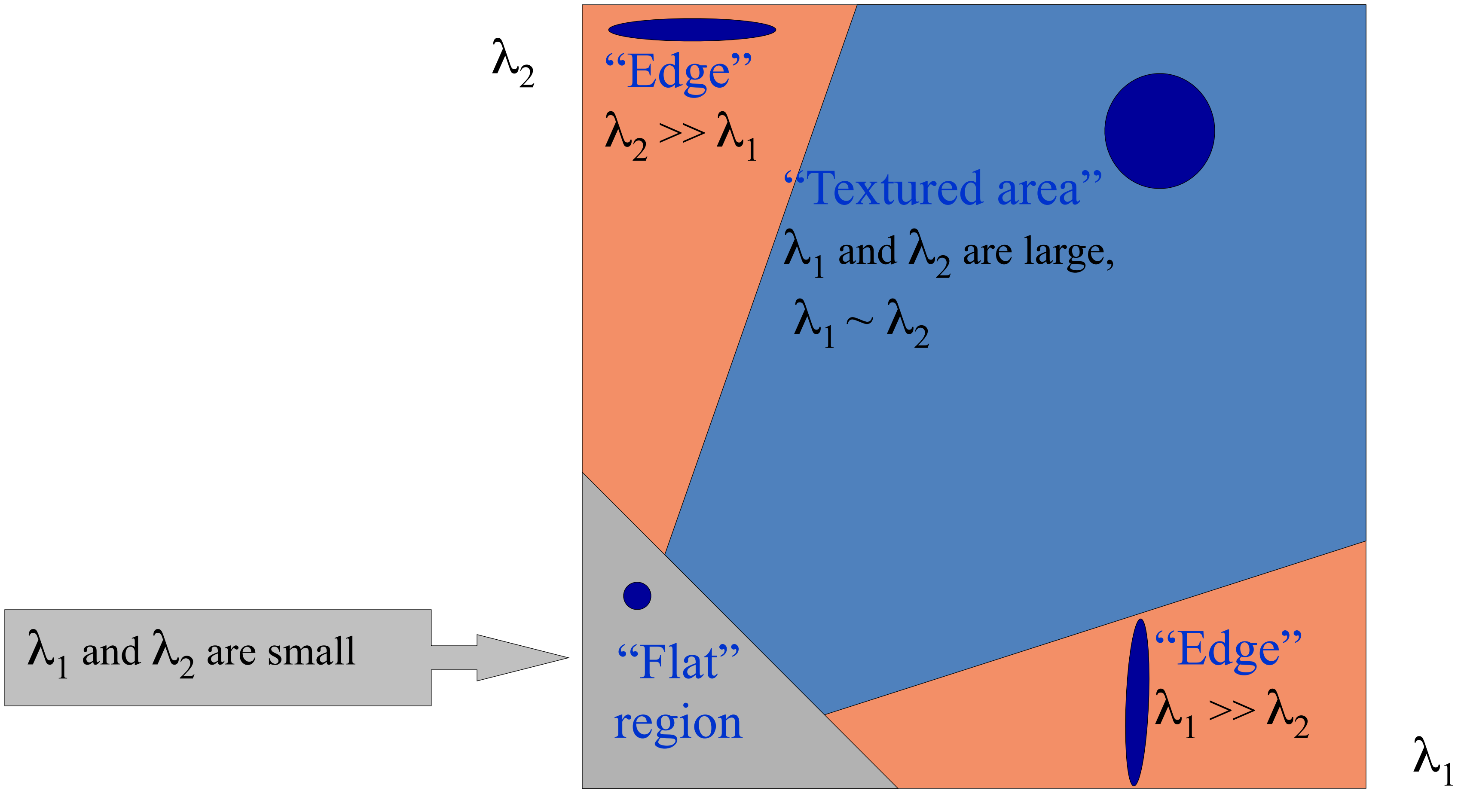
In other words, this quantity is never small

Because $\text{maximize}_{\|w\|^2=1} \; w^T(A^TA)w = \lambda_1$, λ_1 is big

Because $\text{minimize}_{\|w\|^2=1} \; w^T(A^TA)w = \lambda_2$, λ_2 is also big

Interpreting the Eigenvalues

Classification of image points using eigenvalues of $A^T A$



Interpreting the Eigenvalues

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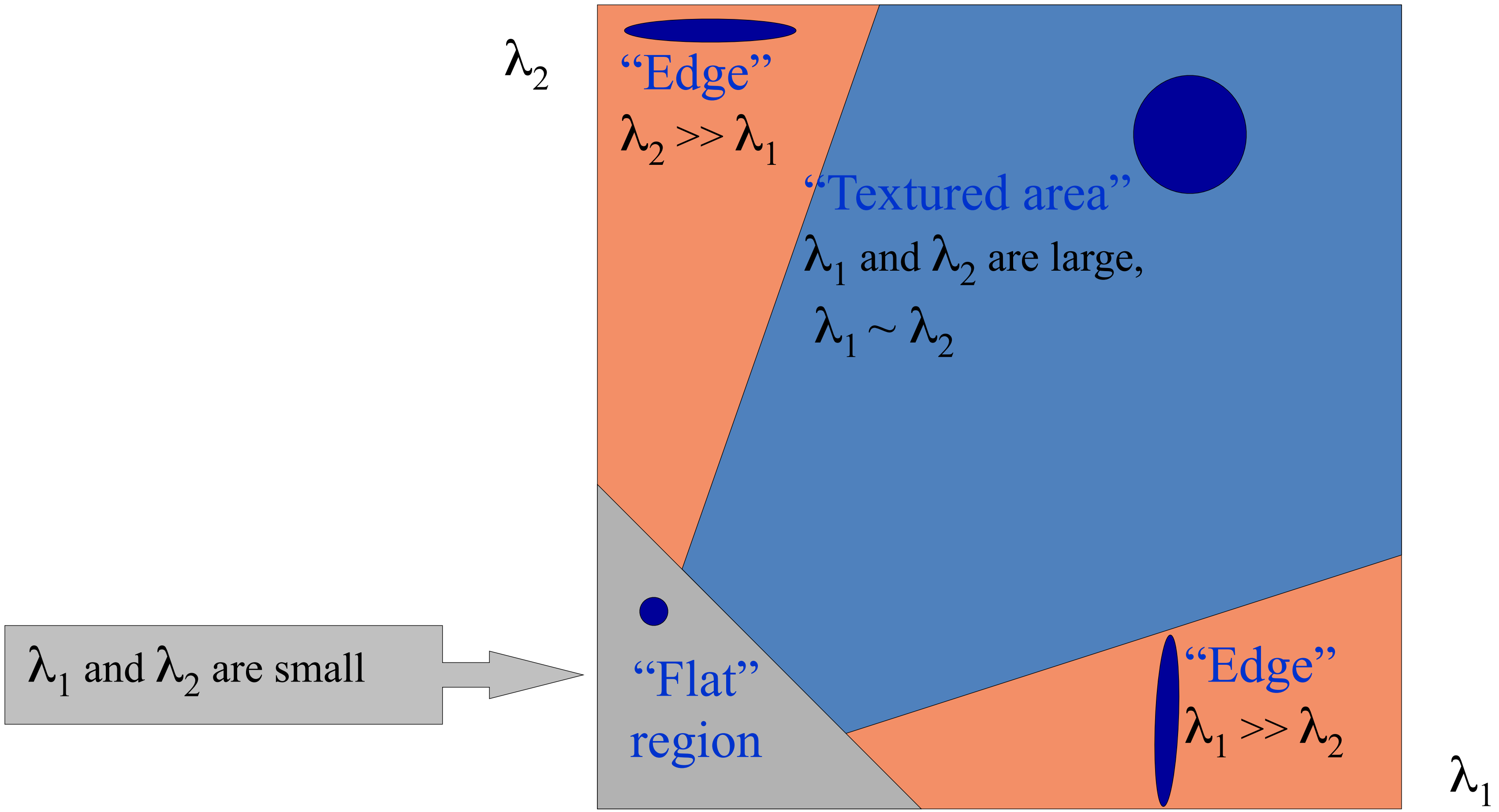
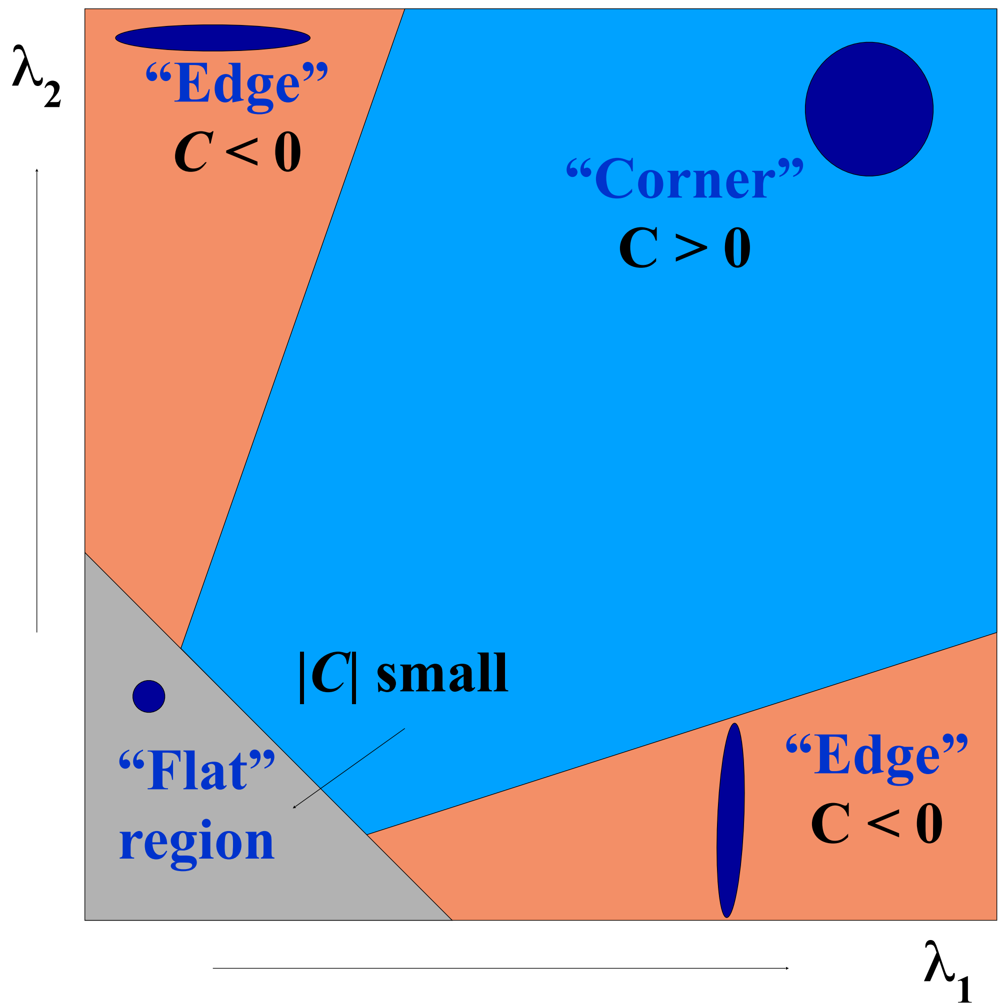


Harris Corner Detector

Cornerness

$$C = \lambda_1 \lambda_2 - \alpha (\lambda_1 + \lambda_2)^2$$

α: constant (0.04 to 0.06)



Harris Corner Detector

Harris Corner Detector

Errors in Lukas-Kanade

- What are the potential causes of errors in this procedure?
  - Suppose $A^T A$ is easily invertible
  - Suppose there is not much noise in the image

- When our assumptions are violated (Taylor expansion fails)
  - Brightness constancy is **not** satisfied
  - The motion is **not** small
  - A point does **not** move like its neighbors
    - window size is too large

* From Khurram Hassan-Shafique CAP5415 Computer Vision 2003